Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 1951–1976
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

The zeta-determinants of Dirac Laplacians with boundary conditions on the smooth, self-adjoint Grassmannian

Yoonweon Lee

Department of Mathematics, Inha University, College of Natural Science, 253, Yonghyun-Dong, Nam-Ku, Incheon, 402-751, Republic of Korea

Received 28 March 2006; received in revised form 9 March 2007; accepted 18 April 2007
Available online 25 April 2007

Abstract

In this paper we describe the difference of log of two zeta-determinants of Dirac Laplacians subject to the Dirichlet boundary condition and a boundary condition on the smooth, self-adjoint Grassmannian $\mathrm{Gr}^*_\infty(D)$ on a compact manifold with boundary. Using this result we obtain the result of Scott and Wojciechowski [S.G. Scott, Zeta determinants on manifolds with boundary, J. Funct. Anal. 192 (2002) 112–185; S.G. Scott, K.P. Wojciechowski, The ζ-determinant and Quillen determinant for a Dirac operator on a manifold with boundary, Geom. Funct. Anal. 10 (2000) 1202–1236] concerning the quotient of two zeta-determinants of Dirac Laplacians with boundary conditions on $\mathrm{Gr}^*_\infty(D)$. We apply these results to the BFK-gluing formula to obtain the gluing formula for the zeta-determinants of Dirac Laplacians with respect to boundary conditions on $\mathrm{Gr}^*_\infty(D)$. We next discuss the zeta-determinants of Dirac Laplacians subject to the Dirichlet or APS boundary condition on a finite cylinder and finally discuss the relative zeta-determinant on a manifold with cylindrical end when the APS boundary condition is imposed on the bottom of the cylinder.

MSC: 58J52; 58J50

Keywords: Zeta- and relative zeta-determinants; BFK-gluing formula; Dirichlet and APS boundary conditions; Smooth self-adjoint Grassmannian; Calderón projector

1. Introduction and results

The zeta-determinants of Laplacians subject to the Dirichlet boundary condition have been studied by many authors in different contexts. For instance, Burghelea, Friedlander and Kappeler [5] proved the gluing formula for the zeta-determinants of Laplacians on a closed manifold with respect to the Dirichlet boundary condition. The relative zeta-determinant of Laplacians on a manifold with cylindrical end was studied by Loya, Park [16] and Müller, Müller [23] independently when the Dirichlet boundary condition is imposed on the bottom of the cylinder. One way of extending these results to the cases of other boundary conditions is to compare the zeta-determinants of Laplacians subject to the Dirichlet boundary condition with the ones subject to given boundary conditions.

In this paper we first describe the difference of the logs of two zeta-determinants of Dirac Laplacians subject to the Dirichlet boundary condition and a boundary condition on the smooth, self-adjoint Grassmannian $\mathrm{Gr}^*_\infty(D)$ on a

E-mail address: ywonlee@inha.ac.kr.

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.04.001

compact manifold with boundary. Using this result we obtain the result of Scott and Wojciechowski [25,26] concerning the quotient of two zeta-determinants of Dirac Laplacians subject to boundary conditions P_1, P_2 on $\mathrm{Gr}^*_\infty(D)$. We next apply these results to the BFK-gluing formula to obtain the gluing formula for the zeta-determinants of Dirac Laplacians with respect to boundary conditions on $\mathrm{Gr}^*_\infty(D)$. In fact, Loya and Park [18,19] have already obtained the same result but their method is different from the one that we present here. Moreover, it is an advantage of this approach to be able to see the relation between the result of this paper and the BFK-gluing formula. Obviously, the Atiyah–Patodi–Singer (APS) boundary condition belongs to $\mathrm{Gr}^*_\infty(D)$ and we discuss the zeta-determinants of Dirac Laplacians subject to the Dirichlet or APS boundary condition on a finite cylinder and finally discuss the relative zeta-determinant on a manifold with cylindrical end when the APS boundary condition is imposed on the bottom of the cylinder.

Now we introduce the basic settings. Let (M, g) be a compact oriented $(m+1)$-dimensional Riemannian manifold $(m > 0)$ with boundary Y and $E \to M$ be a Clifford module bundle. Choose a collar neighborhood N of Y which is diffeomorphic to $[0, 1) \times Y$. We assume that the metric g is a product one on N and the bundle E has the product structure on N, which means that $E|_N = p^*E|_Y$, where $p : [0, 1) \times Y \to Y$ is the canonical projection. Suppose that D_M is a compatible Dirac operator acting on smooth sections of E. We assume that D_M has the following form on N:

$$D_M = G(\partial_u + B),$$

where $G : E|_Y \to E|_Y$ is a bundle automorphism, ∂_u is the inward normal derivative to Y on N and B is a Dirac operator on Y. We further assume that G and B are independent of the normal coordinate u and satisfy

$$G^* = -G, \qquad G^2 = -I, \qquad B^* = B, \qquad GB = -BG,$$
$$\dim(\ker(G - i) \cap \ker B) = \dim(\ker(G + i) \cap \ker B). \tag{1.1}$$

Then we have, on N, the Dirac Laplacian

$$D_M^2 = -\partial_u^2 + B^2.$$

We next introduce the boundary conditions on Y. The Dirichlet boundary condition on Y is defined by the restriction map $\gamma_0 : C^\infty(M) \to C^\infty(Y)$, $\gamma_0(\phi) = \phi|_Y$, and the realization D^2_{M,γ_0} is defined to be the operator D_M^2 with the following domain:

$$\mathrm{Dom}(D^2_{M,\gamma_0}) = \{\phi \in C^\infty(E) \mid \phi|_Y = 0\}.$$

Then D^2_{M,γ_0} is an invertible operator by the unique continuation property of D_M (cf. [2]).

The APS boundary condition $\Pi_>$ (or $\Pi_<$) is defined to be the orthogonal projection to the space spanned by positive (or negative) eigensections of B. If $\ker B \neq \{0\}$, we need an extra condition to obtain a self-adjoint operator, say, a unitary involution on $\ker B$ anticommuting with G. Suppose that $\sigma : \ker B \to \ker B$ is a unitary operator satisfying

$$\sigma G = -G\sigma, \qquad \sigma^2 = \mathrm{Id}_{\ker B}.$$

We put $\sigma^\pm = \frac{I\pm\sigma}{2}$ and define $\Pi_{<,\sigma^-}$, $\Pi_{>,\sigma^+}$ as

$$\Pi_{<,\sigma^-} = \Pi_< + \frac{1}{2}(I - \sigma)|_{\ker B}, \qquad \Pi_{>,\sigma^+} = \Pi_> + \frac{1}{2}(I + \sigma)|_{\ker B}.$$

Then the realizations $D_{M,\Pi_{<,\sigma^-}}$ and $D^2_{M,\Pi_{<,\sigma^-}}$ are defined by D_M and D_M^2 with the following domains:

$$\mathrm{Dom}(D_{M,\Pi_{<,\sigma^-}}) = \{\phi \in C^\infty(E) \mid \Pi_{<,\sigma^-}(\phi|_Y) = 0\},$$
$$\mathrm{Dom}(D^2_{M,\Pi_{<,\sigma^-}}) = \{\phi \in C^\infty(E) \mid \Pi_{<,\sigma^-}(\phi|_Y) = 0, \Pi_{>,\sigma^+}((\partial_u + B)\phi)|_Y = 0\}.$$

$D_{M,\Pi_{>,\sigma^+}}$ and $D^2_{M,\Pi_{>,\sigma^+}}$ are defined similarly.

As a generalization of the APS boundary condition we introduce the self-adjoint Grassmannian $\mathrm{Gr}^*(D)$, which is the set of all orthogonal pseudodifferential projections P such that

$$-GPG = \mathrm{Id} - P, \quad P - \Pi_> \text{ is a classical pseudodifferential operator of order } -1.$$

As a dense subset of $\mathrm{Gr}^*(D)$, we define $\mathrm{Gr}^*_\infty(D)$ by

$$\mathrm{Gr}^*_\infty(D) = \{P \in \mathrm{Gr}^*(D) \mid P - \Pi_> \text{ is a smoothing operator }\}. \tag{1.2}$$

Then Wojciechowski [28] showed that $\eta_{D_P}(s)$ and $\zeta_{D_P^2}(s)$ for $P \in \mathrm{Gr}^*_\infty(D)$ have regular values at $s = 0$. Clearly, $\Pi_{>,\sigma^+}$ belongs to $\mathrm{Gr}^*_\infty(D)$. The Calderón projector $\mathfrak{C}$ is defined to be the orthogonal projection from $L^2(E|_Y)$ onto $\overline{\{\phi|_Y \mid D_M(\phi) = 0\}}$, the Cauchy data space. Then $\mathfrak{C}$ is known to be an element of $\mathrm{Gr}^*_\infty(D)$ by Scott [24] and Grubb [9]. The realization $D^2_{M,P}$ is defined to be the operator D^2_M with the following domain:

$$\mathrm{Dom}(D^2_{M,P}) = \left\{\phi \in C^\infty(M) \mid P\gamma_0\phi = 0, (I - P)\gamma_0(\partial_u + B)\phi = 0\right\}. \tag{1.3}$$

The purpose of this paper is to describe the relative zeta-determinant $\log \mathrm{Det}\, D^2_{M,P} - \log \mathrm{Det}\, D^2_{M,\gamma_0}$ and discuss some of its applications including the gluing formula for the zeta-determinants of Dirac Laplacians.

To describe the main result we define $Q : C^\infty(Y) \to C^\infty(Y)$ as follows. For $f \in C^\infty(Y)$ there exists a unique section $\phi \in C^\infty(M)$ satisfying $D^2_M\phi = 0$, $\phi|_Y = f$. Then we define

$$Q(f) = -(\partial_u\phi)|_Y. \tag{1.4}$$

The Green formula shows that $Q - B$ is a non-negative operator with $\ker(Q - B) = \mathrm{Im}\,\mathfrak{C}$, the Cauchy data space (Lemma 2.5). We regard $(I - P)(Q - B)(I - P)$ as an operator on $\mathrm{Im}\,(I - P)$, i.e.,

$$(I - P)(Q - B)(I - P) : C^\infty(Y) \cap \mathrm{Im}\,(I - P) \to C^\infty(Y) \cap \mathrm{Im}\,(I - P).$$

Using the fact that $Q - |B|$ [13] and $P - \Pi_>$ are smoothing operators, we can show that the zeta-determinant of $(I - P)(Q - B)(I - P)$ is well-defined (2.14). It is not difficult to show that $\ker\,(I - P)(Q - B)(I - P) = \{\psi|_Y \mid \psi \in \ker D_{M,P}\}$ (Lemma 2.5). Let $\{h_1, h_2, \ldots, h_q\}$ be an orthonormal basis for $\ker\,((I - P)(Q - B)(I - P))$, $q = \dim\ker D_{M,P}$. Then there exist $\psi_1, \psi_2, \ldots, \psi_q$ such that

$$D_{M,P}\psi_i = 0, \qquad \psi_i|_Y = h_i.$$

We define a $q \times q$ positive definite Hermitian matrix $V_{M,P}$ by

$$V_{M,P} = \left(\langle\psi_i, \psi_j\rangle_M\right)_{1\le i,j\le q}. \tag{1.5}$$

If $\mathfrak{P}$ is an invertible elliptic operator of order >0 with discrete spectrum $\{\mu_j \mid j = 1, 2, 3, \ldots\}$, we define the zeta function $\zeta_{\mathfrak{P}}(s) = \sum_{\mu_j\in\mathrm{Spec}(\mathfrak{P})} \mu_j^{-s}$ and the zeta-determinant $\mathrm{Det}\,\mathfrak{P}$ by $\mathrm{e}^{-\zeta'_{\mathfrak{P}}(0)}$. If $\mathfrak{P}$ has a non-trivial kernel, we define the modified zeta-determinant $\mathrm{Det}^*\,\mathfrak{P}$ by

$$\mathrm{Det}^*\,\mathfrak{P} := \mathrm{Det}(\mathfrak{P} + \mathrm{pr}_{\ker\mathfrak{P}}).$$

Similarly, if α is a trace class operator, we define the modified Fredholm determinant by

$$\det{}^*_{\mathrm{Fr}}(I + \alpha) = \det{}_{\mathrm{Fr}}(I + \alpha + \mathrm{pr}_{\ker(I+\alpha)}).$$

Equivalently, $\mathrm{Det}^*\,\mathfrak{P}$ and $\det^*_{\mathrm{Fr}}(I + \alpha)$ are the determinants of $\mathfrak{P}$ and $I + \alpha$ when restricted to the orthogonal complements of $\ker\mathfrak{P}$ and $\ker(I + \alpha)$, respectively.

Then the following is the main result of this paper.

Theorem 1.1. *Suppose that (M, g) is a compact Riemannian manifold with boundary Y having the product structure near the boundary and D_M is a compatible Dirac operator which has the form (1.1) near the boundary. Then for $P \in \mathrm{Gr}^*_\infty(D)$ and the Dirichlet boundary condition γ_0 on Y, we have the following equality:*

$$\log \mathrm{Det}^*\, D^2_{M,P} - \log \mathrm{Det}\, D^2_{M,\gamma_0} = \log\det V_{M,P} + \log \mathrm{Det}^*\,((I - P)(Q - B)(I - P)),$$

where $((I - P)(Q - B)(I - P))$ is considered to be an operator defined on $\mathrm{Im}(I - P)$.

Remark. (1) We take the negative real axis as a branch cut for the logarithm.

(2) If we parametrize the collar neighborhood N by $(-1, 0] \times Y$ with the boundary $\{0\} \times Y$ and write the Dirac operator D_M on N by $D_M = G(\partial_u + B)$ with ∂_u the outward unit normal derivative, $Q(f)$ is defined by

$$Q(f) := (\partial_u \phi)|_Y, \quad \text{where } D_M^2 \phi = 0 \text{ and } \phi|_Y = f. \tag{1.6}$$

Then $(Q + B)$ is a non-negative operator and in this case Theorem 1.1 can be written as follows:

$$\log \mathrm{Det}^* D^2_{M,I-P} - \log \mathrm{Det}\, D^2_{M,\gamma_0} = \log \det V_{M,I-P} + \log \mathrm{Det}^* \left(P(Q+B)P\right). \tag{1.7}$$

(3) Even if the boundary of M consists of two components Y and Z, Theorem 1.1 still holds as long as M has the product structures near Y and a boundary condition $\mathfrak{B}$ is imposed on Z so that $D^2_{M,\mathfrak{B},\gamma_0}$ is an invertible operator. For example, if $\mathfrak{B}$ is the Dirichlet boundary condition on Z, both $D^2_{M,\mathfrak{B},P}$ and $D^2_{M,\mathfrak{B},\gamma_0}$ are invertible operators. In this case, Q is defined as follows. For $f \in C^\infty(Y)$, choose $\phi \in C^\infty(M)$ such that $D_M^2\phi = 0$, $\phi|_Z = 0$ and $\phi|_Y = f$. Then $Q(f) := -(\partial_u\phi)|_Y$. Since the term $\log\det V_{M,P}$ does not appear in this case, Theorem 1.1 is written as

$$\log \mathrm{Det}\, D^2_{M,\mathfrak{B},P} - \log \mathrm{Det}\, D^2_{M,\mathfrak{B},\gamma_0} = \log \mathrm{Det}\left((I-P)(Q-B)(I-P)\right). \tag{1.8}$$

Since G is a bundle automorphism with $G^2 = -I$, $E|_Y$ splits into $\pm i$-eigenspaces $E_Y^\pm$ of G, say, $E|_Y = E_Y^+ \oplus E_Y^-$, and the Dirac operator D_M is written as

$$D_M = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left(\partial_u + \begin{pmatrix} 0 & B^- \\ B^+ & 0 \end{pmatrix} \right),$$

where $B^\pm : C^\infty(E_Y^\pm) \to C^\infty(E_Y^\mp)$ and $\left(B^\pm\right)^* = B^\mp$. Then there exists the unitary operator $K : L^2(Y, E_Y^+) \to L^2(Y, E_Y^-)$ satisfying $\mathrm{Im}\,\mathfrak{C} = \mathrm{graph}(K)$. For $P \in \mathrm{Gr}^*_\infty(D)$, there exists a unitary operator $T : L^2(Y, E_Y^+) \to L^2(Y, E_Y^-)$ such that

$$\mathrm{Im}\, P = \mathrm{graph}(T), \quad T = K + \text{ a smoothing operator}. \tag{1.9}$$

As the first application of Theorem 1.1 we obtain the following result, which was proved earlier by Scott and Wojciechowski [25,26].

Theorem 1.2. *Suppose that P is a pseudodifferential projection in $\mathrm{Gr}^*_\infty(D)$. Then,*

$$\frac{\mathrm{Det}^* D^2_{M,P}}{\mathrm{Det}\, D^2_{M,\mathfrak{C}}} = \left(\det V_{M,P}\right)^2 \cdot \left| \det{}^*_{\mathrm{Fr}} \left(\frac{1}{2}\left(I + T^{-1}K\right)\right)\right|^2.$$

We next apply Theorems 1.1 and 1.2 to the BFK-gluing formula for the zeta-determinants of Dirac Laplacians. Let $(\widetilde{M}, \widetilde{g})$ be a closed Riemannian manifold and Y be a hypersurface of $\widetilde{M}$ such that $\widetilde{M} - Y$ has two components. We denote by M_1, M_2 the closure of each component. We choose a collar neighborhood of Y which is diffeomorphic to $(-1, 1) \times Y$ and assume that $\widetilde{g}$ is a product metric on N. Let $\widetilde{E} \to \widetilde{M}$ be a Clifford module bundle having the product structure on N and $\widetilde{D}$ be a compatible Dirac operator acting on smooth sections of $\widetilde{E}$ which has the form, on N, $\widetilde{D} = G(\partial_u + B)$ satisfying (1.1) as before. We denote by D_{M_1}, D_{M_2} the restrictions of $\widetilde{D}$ to M_1, M_2 and by γ_0 the Dirichlet boundary condition on Y. Suppose that $\{h_1, h_2, \ldots, h_q\}$ is an orthonormal basis for $\left(\ker \widetilde{D}\right)|_Y := \{\Phi|_Y \mid \widetilde{D}\Phi = 0\}$, where $q = \dim\ker\widetilde{D}$. Then there exist $\Phi_1, \ldots, \Phi_q$ in $\ker\widetilde{D}$ with $\Phi_i|_Y = h_i$. We define a positive definite Hermitian matrix A_0 by

$$A_0 = (a_{ij}), \quad \text{where } a_{ij} = \langle \Phi_i, \Phi_j \rangle_{\widetilde{M}}. \tag{1.10}$$

Now the BFK-gluing formula can be stated as follows: (cf. [5,13]).

$$\log \mathrm{Det}^* \widetilde{D}^2 - \log \mathrm{Det}\, D^2_{M_1,\gamma_0} - \log \mathrm{Det}\, D^2_{M_2,\gamma_0} = -\log 2 \cdot (\zeta_{B^2}(0) + l) + \log\det A_0 + \log \mathrm{Det}^*(Q_1 + Q_2), \tag{1.11}$$

where $l = \dim\ker B$ and Q_1 is defined by (1.6), Q_2 by (1.4). Theorems 1.1 and 1.2 together with (1.11) lead to the following result, which is the main motivation for Theorem 1.1.

Theorem 1.3. *Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ be Calderón projectors for D_{M_1}, D_{M_2} and P_1, P_2 be orthogonal pseudodifferential projections belonging to $\mathrm{Gr}^*_\infty(D_{M_1})$, $\mathrm{Gr}^*_\infty(D_{M_2})$, respectively. Suppose that for $i = 1, 2$, $K_i, T_i : L^2(Y, E^+_Y) \to L^2(Y, E^-_Y)$ are unitary maps such that $\mathrm{graph}(K_i) = \mathrm{Im}\, \mathfrak{C}_i$ and $\mathrm{graph}(T_i) = \mathrm{Im}\, P_i$. Then the following equalities hold:*

$$(1) \log \mathrm{Det}^* \widetilde{D}^2 - \log \mathrm{Det}\, D^2_{M_1,\mathfrak{C}_1} - \log \mathrm{Det}\, D^2_{M_2,\mathfrak{C}_2} = -\log 2 \cdot (\zeta_{B^2}(0) + l) + 2 \log \det A_0 + 2 \log \left| \det{}^*_{\mathrm{Fr}} \left(\frac{1}{2} \left(I - K_1^{-1} K_2 \right) \right) \right| .$$

$$(2) \log \mathrm{Det}^* \widetilde{D}^2 - \log \mathrm{Det}^*\, D^2_{M_1,P_1} - \log \mathrm{Det}^*\, D^2_{M_2,P_2} = -\log 2 \cdot (\zeta_{B^2}(0) + l) + 2 \log \det A_0 - 2 \sum_{i=1}^{2} \log \det V_{M_i,P_i} + 2 \log \left| \det{}^*_{\mathrm{Fr}} \left(\frac{1}{2} \left(I - K_1^{-1} K_2 \right) \right) \right| - 2 \sum_{i=1}^{2} \log \left| \det{}^*_{\mathrm{Fr}} \left(\frac{1}{2} \left(I + T_i^{-1} K_i \right) \right) \right| .$$

Remark. The result of Theorem 1.3 was obtained earlier by Loya and Park [18,19] in a different way.

We next apply Theorem 1.1 to Laplacians on a cylinder. Let $N_{0,r} := [0, r] \times Y$ and denote by $(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\Pi_{<,\sigma^-}}$ ($(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\gamma_r}$) the Dirac Laplacian subject to the Dirichlet boundary condition on Y_0 and $\Pi_{<,\sigma^-}$ on $Y_r := \{r\} \times Y$ (the Dirichlet boundary condition γ_0, γ_r on Y_0, Y_r). Then $(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\gamma_r}$ and $(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\Pi_{<,\sigma^-}}$ are invertible operators and Q_1 is expressed as (cf. [14])

$$Q_1 = \frac{1}{r} \mathrm{pr}_{\ker B} + |B| + \frac{2|B| \mathrm{e}^{-r|B|}}{\mathrm{e}^{r|B|} - \mathrm{e}^{-r|B|}} \mathrm{pr}_{(\ker B)^\perp}. \tag{1.12}$$

In this case Theorem 1.1 is stated as follows, which was obtained earlier in [14,15].

Corollary 1.4. *Suppose that $l = \dim \ker B$ and $N_{0,r} = [0, r] \times Y$. Then,*

$$\begin{aligned} &\log \mathrm{Det}(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\Pi_{<,\sigma^-}} - \log \mathrm{Det}(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\gamma_r} \\ &\quad = \log \mathrm{Det}(-\partial_u^2 + B^2)_{N_{0,r},\Pi_{>,\sigma^+},\gamma_r} - \log \mathrm{Det}(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\gamma_r} \\ &\quad = -\frac{l}{2} \cdot \log r + \frac{1}{2} \log 2 \cdot \zeta_{B^2}(0) + \frac{1}{4} \log \mathrm{Det}^* B^2 + \frac{1}{2} \log \det{}_{\mathrm{Fr}} \left(I + \frac{\mathrm{e}^{-r|B|}}{\mathrm{e}^{r|B|} - \mathrm{e}^{-r|B|}} \mathrm{pr}_{(\ker B)^\perp} \right). \end{aligned}$$

We next consider the Dirac Laplacian $(-\partial_u^2 + B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}}$ on $N_{0,r}$ with the boundary conditions $\Pi_{>,\tau^+}$ on Y_0 and $\Pi_{<,\sigma^-}$ on Y_r, where σ and τ are unitary involutions on $\ker B$ anticommuting with G. Then it is not difficult to see that

$$\ker(-\partial_u^2 + B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} = \{ f \in C^\infty(Y) \mid f \in (\mathrm{Im}\, \tau^- \cap \mathrm{Im}\, \sigma^+) \}. \tag{1.13}$$

We also introduce the boundary condition $(\partial_u + |B|)$ on Y_r and denote by $(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)}$ the Dirac Laplacian subject to the Dirichlet condition γ_0 on Y_0 and $(\partial_u + |B|)$ on Y_r, i.e.

$$\mathrm{Dom}((-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)}) = \{ \phi \in C^\infty(N_{0,r}) \mid \phi|_{Y_0} = 0, ((\partial_u + |B|)\phi)\,|_{Y_r} = 0 \}.$$

Then $(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)}$ is an invertible operator. To describe the next result we introduce a constant α_1 as follows. We first consider the asymptotic expansion of the heat kernel of B^2. We note that $\dim Y = m$. Then, as $t \to 0^+$,

$$\mathrm{Tr}\, \mathrm{e}^{-tB^2} \sim \sum_{j=0}^{\infty} b_j t^{-\frac{m}{2}+j}.$$

This series shows that $\zeta_{B^2}(s)$ is analytic at $s = -\frac{1}{2}$ if m is even. However, if m is odd, $\zeta_{B^2}(s)$ has a simple pole at $s = -\frac{1}{2}$. We define α_1 by

$$\alpha_1 = \begin{cases} \zeta_{B^2}\left(-\frac{1}{2}\right), & \text{if } m \text{ is even} \\ \frac{\mathrm{d}}{\mathrm{d}s}\left(s \cdot \zeta_{B^2}\left(s - \frac{1}{2}\right)\right)\Big|_{s=0} + \frac{1}{\sqrt{\pi}}(\log 2 - 1) \cdot b_{\frac{m+1}{2}}, & \text{if } m \text{ is odd.} \end{cases} \tag{1.14}$$

Then we have the following result.

Theorem 1.5. *Suppose that* $l = \dim\ker B$ *and* $k_+ = \dim(\operatorname{Im}\sigma^+ \cap \operatorname{Im}\tau^-)$. *Then:*

(*1*) $\log\operatorname{Det}^*(-\partial_u^2 + B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} = \alpha_1 \cdot r + 2k_+ \log r + \log 2 \cdot (\zeta_{B^2}(0) + l) + \log\left|\det{}^*\left(\frac{\sigma+\tau}{2}\right)\right|,$

where $\det^*\left(\frac{\sigma+\tau}{2}\right) = \det\left(\frac{\sigma+\tau}{2} + \mathrm{pr}_{\ker(\sigma+\tau)}\right)$.

(*2*) $\log\operatorname{Det}(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)} = \alpha_1 \cdot r + \log 2 \cdot (\zeta_{B^2}(0) + l)$.

Remark. The first equality in Theorem 1.5 was proved first by Loya and Park in [17].

Finally, we are going to apply Theorem 1.1 to the relative zeta-determinant on a manifold with cylindrical end studied by Müller, Müller in [23] and Loya, Park in [16]. Let $M_{1,\infty} = M_1 \cup_Y [0,\infty) \times Y$ and $N_{0,\infty} = [0,\infty) \times Y$. We denote by $D_{M_{1,\infty}}$ the extension of D_{M_1} to $M_{1,\infty}$ and by $(-\partial_u^2 + B^2)_{N_{0,\infty},\gamma_0}$ the Dirac Laplacian on $N_{0,\infty}$ subject to the Dirichlet boundary condition on $\{0\} \times Y$. Let μ_1 be the smallest positive eigenvalue of B. Then the scattering theory for a Dirac operator on a manifold with cylindrical end [11,21] shows that $D_{M_{1,\infty}}$ determines a regular one-parameter family of unitary operators $C(v)$, called on-shell scattering operators, with $v \in \mathbb{R}$, $|v| < \mu_1$, which act on $\ker B$ and satisfy

$$C(v)C(-v) = I, \qquad C(v)G = -GC(v).$$

They showed independently in [23,16] that for $l = \dim\ker B$,

$$\begin{aligned} &\log\operatorname{Det}\left(D^2_{M_{1,\infty}}, (-\partial_u^2 + B^2)_{N_{0,\infty},\gamma_0}\right) - \log\operatorname{Det}(D^2_{M_1,\gamma_0}) \\ &\quad = -\log 2 \cdot (\zeta_{B^2}(0) + l) + \log\operatorname{Det}^*(Q_1 + |B|) - \log\det A_1, \end{aligned} \tag{1.15}$$

where A_1 is a positive definite Hermitian matrix defined as follows. Let $\{\psi_1, \ldots, \psi_{q'}\}$ be an orthonormal basis for the space of L^2-solutions of $D_{M_{1,\infty}}$ on $M_{1,\infty}$ and $\{f_1, \ldots, f_{\frac{l}{2}}\}$ be an orthonormal basis for $\operatorname{Im} C(0)^+$, the space of the limiting values of the extended L^2-solutions of $D_{M_{1,\infty}}$. We put $\psi_{q'+j} = \frac{1}{2}E(f_j, 0)$ for $1 \le j \le \frac{l}{2}$, where $\frac{1}{2}E(f_j, 0)$ is an extended L^2-solution of $D_{M_{1,\infty}}$ on $M_{1,\infty}$ whose limiting value is f_j (see [21] or [23] for notation and definitions). Then we define

$$A_1 = (a_{ij}), \quad \text{where } a_{ij} = \langle \psi_i|_Y, \psi_j|_Y \rangle_Y,\ 1 \le i, j \le q' + \frac{l}{2}. \tag{1.16}$$

Applying Theorem 1.1 to (1.15), we have the following result.

Theorem 1.6.

$$\begin{aligned} &\log\operatorname{Det}\left(D^2_{M_{1,\infty}}, (-\partial_u^2 + B^2)_{N_{0,\infty},\Pi_{>,\tau^+}}\right) - \log\operatorname{Det}(D^2_{M_1,\Pi_{<,\sigma^-}}) = -\log 2 \cdot (\zeta_{B^2}(0) + l) - 2\log\det A_1 \\ &\quad - 2\log\det V_{M_1,\Pi_{<,\sigma^-}} + 2\log\left|\det{}^*_{\mathrm{Fr}}\left(\frac{1}{2}\left(I - K_1^{-1}T_0\right)\right)\right| - 2\log\left|\det{}^*_{\mathrm{Fr}}\left(\frac{1}{2}\left(I - K_1^{-1}T_{\sigma^+}\right)\right)\right|, \end{aligned}$$

where $\operatorname{graph}(T_0) = \operatorname{Im}\Pi_{>,C(0)^+}$ *and* $\operatorname{graph}(T_{\sigma^+}) = \operatorname{Im}\Pi_{>,\sigma^+}$.

Remark. Lemma 5.1 in Section 5 shows that the left hand side of the above equality does not depend on the choice of a unitary involution τ anticommuting with G.

2. The proof of Theorem 1.1

In this section we are going to prove Theorem 1.1 by using the method used in [5,6,8]. Let P be an orthogonal pseudodifferential projection in $\mathrm{Gr}^*_\infty(D)$ and ν be a positive integer $> \frac{m}{2}$ with $m+1 = \dim M$. Then for $\lambda > 0$, both $(D^2_{M,P}+\lambda)^{-\nu}$ and $(D^2_{M,\gamma_0}+\lambda)^{-\nu}$ are trace class operators. Taking the derivative ν times with respect to λ,

$$\frac{\mathrm{d}^\nu}{\mathrm{d}\lambda^\nu}\left\{\log \mathrm{Det}\left(D^2_{M,P}+\lambda\right) - \log \mathrm{Det}\left(D^2_{M,\gamma_0}+\lambda\right)\right\} = \mathrm{Tr}\left\{\frac{\mathrm{d}^{\nu-1}}{\mathrm{d}\lambda^{\nu-1}}\left(\left(D^2_{M,P}+\lambda\right)^{-1} - \left(D^2_{M,\gamma_0}+\lambda\right)^{-1}\right)\right\}. \tag{2.1}$$

We introduce the Poisson operator for the Dirichlet condition $P_{\gamma_0}(\lambda) : C^\infty(Y) \to C^\infty(M)$, which is characterized as follows. For any $f \in C^\infty(Y)$,

$$\left(D^2_M+\lambda\right) P_{\gamma_0}(\lambda) f = 0, \qquad \gamma_0 P_{\gamma_0}(\lambda) f = f. \tag{2.2}$$

Then we have

$$\left(D^2_{M,P}+\lambda\right)^{-1} - \left(D^2_{M,\gamma_0}+\lambda\right)^{-1} = P_{\gamma_0}(\lambda)\gamma_0\left(D^2_{M,P}+\lambda\right)^{-1}. \tag{2.3}$$

Combining (2.1) with (2.3) leads to

$$\begin{aligned}\frac{\mathrm{d}^\nu}{\mathrm{d}\lambda^\nu}&\left\{\log \mathrm{Det}\left(D^2_{M,P}+\lambda\right) - \log \mathrm{Det}\left(D^2_{M,\gamma_0}+\lambda\right)\right\} = \mathrm{Tr}\left\{\frac{\mathrm{d}^{\nu-1}}{\mathrm{d}\lambda^{\nu-1}}\left(P_{\gamma_0}(\lambda)\gamma_0\left(D^2_{M,P}+\lambda\right)^{-1}\right)\right\} \\ &= \mathrm{Tr}\left\{\frac{\mathrm{d}^{\nu-1}}{\mathrm{d}\lambda^{\nu-1}}\left(\gamma_0\left(D^2_{M,P}+\lambda\right)^{-1} P_{\gamma_0}(\lambda)\right)\right\} \\ &= \mathrm{Tr}\left\{\frac{\mathrm{d}^{\nu-1}}{\mathrm{d}\lambda^{\nu-1}}\left((I-P)\gamma_0\left(D^2_{M,P}+\lambda\right)^{-1} P_{\gamma_0}(\lambda)(I-P)\right)\right\}.\end{aligned} \tag{2.4}$$

According to the method suggested in [8], we define $Q(\lambda) : C^\infty(Y) \to C^\infty(Y)$ by

$$Q(\lambda) = -\gamma_0 \partial_u P_{\gamma_0}(\lambda),$$

and define $R_P(\lambda) : C^\infty(Y) \to C^\infty(Y)$ by

$$R_P(\lambda) = (I-P)(Q(\lambda)-B)(I-P) + P|B|P + \mathrm{pr}_{(\ker B \,\cap\, \mathrm{Im}\, P)}.$$

Then $R_P(\lambda)$ is a positive definite, elliptic ΨDO (cf. Lemma 2.5). Taking the derivative of $R_P(\lambda)$ with respect to λ, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} R_P(\lambda) = -(I-P)\gamma_0\partial_u\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}P_{\gamma_0}(\lambda)\right)(I-P). \tag{2.5}$$

Lemma 2.1.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}P_{\gamma_0}(\lambda) = -\left(D^2_{M,\gamma_0}+\lambda\right)^{-1} P_{\gamma_0}(\lambda).$$

Proof. Taking the derivative in (2.2) with respect to λ, we have

$$\left(D^2_M+\lambda\right)\frac{\mathrm{d}}{\mathrm{d}\lambda}P_{\gamma_0}(\lambda) = -P_{\gamma_0}(\lambda), \qquad \gamma_0\frac{\mathrm{d}}{\mathrm{d}\lambda}P_{\gamma_0}(\lambda) = 0,$$

which implies the result. □

Since $(I-P)\gamma_0(\partial_u+B)(D^2_{M,P}+\lambda)^{-1}=0$ (cf. (1.3)), Eq. (2.5) and Lemma 2.1 lead to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}R_P(\lambda) &= (I-P)\gamma_0\partial_u\left(D^2_{M,\gamma_0}+\lambda\right)^{-1}P_{\gamma_0}(\lambda)(I-P)\\ &= -(I-P)\gamma_0\,(\partial_u+B)\left(\left(D^2_{M,P}+\lambda\right)^{-1}-\left(D^2_{M,\gamma_0}+\lambda\right)^{-1}\right)P_{\gamma_0}(\lambda)(I-P)\\ &= -(I-P)\gamma_0\,(\partial_u+B)\,P_{\gamma_0}(\lambda)\gamma_0\left(D^2_{M,P}+\lambda\right)^{-1}P_{\gamma_0}(\lambda)(I-P)\\ &= -(I-P)\left(\gamma_0\cdot\partial_u\cdot P_{\gamma_0}(\lambda)+B\right)(I-P)\gamma_0\left(D^2_{M,P}+\lambda\right)^{-1}P_{\gamma_0}(\lambda)(I-P)\\ &= -(I-P)\,(-Q(\lambda)+B)\,(I-P)\gamma_0\left(D^2_{M,P}+\lambda\right)^{-1}P_{\gamma_0}(\lambda)(I-P)\\ &= R_P(\lambda)(I-P)\gamma_0\left(D^2_{M,P}+\lambda\right)^{-1}P_{\gamma_0}(\lambda)(I-P),\end{aligned}\tag{2.6}$$

which shows that

$$R_P(\lambda)^{-1}\frac{\mathrm{d}}{\mathrm{d}\lambda}R_P(\lambda)=(I-P)\gamma_0\left(D^2_{M,P}+\lambda\right)^{-1}P_{\gamma_0}(\lambda)(I-P).\tag{2.7}$$

Combining (2.4) with (2.7), we have

$$\begin{aligned}\frac{\mathrm{d}^\nu}{\mathrm{d}\lambda^\nu}\left\{\log\mathrm{Det}\left(D^2_{M,P}+\lambda\right)-\log\mathrm{Det}\left(D^2_{M,\gamma_0}+\lambda\right)\right\} &= \mathrm{Tr}\left\{\frac{\mathrm{d}^{\nu-1}}{\mathrm{d}\lambda^{\nu-1}}\left(R_P(\lambda)^{-1}R_P(\lambda)\right)\right\}\\ &= \frac{\mathrm{d}^\nu}{\mathrm{d}\lambda^\nu}\log\mathrm{Det}\,R_P(\lambda).\end{aligned}\tag{2.8}$$

Now we discuss the well-definedness of the zeta-determinants of $P|B|P$ and $(I-P)\,(Q(\lambda)-B)\,(I-P)$. Since $P|B|P$ and $(I-P)\,(Q(\lambda)-B)\,(I-P)$ are not elliptic operators, we should be careful to say the zeta-determinants. Using the fact that $P-\Pi_{>,\sigma^+}$ is a smoothing operator, we have

$$P|B|P+(I-P)|B|(I-P)=|B|+\text{ a smoothing operator,}$$

which shows that $(P|B|P+(I-P)|B|(I-P))$ is an elliptic operator on Y. Furthermore, the spectrum of $(P|B|P+(I-P)|B|(I-P))$ is the union of the spectrum of $P|B|P$ on $\mathrm{Im}\,P$ and the spectrum of $(I-P)|B|(I-P)$ on $\mathrm{Im}(I-P)$ because of the complementary orthogonal projections P and $I-P$. If $\phi_1,\phi_2,\ldots$ are all eigensections of $P|B|P$ on $\mathrm{Im}\,P$ with the corresponding eigenvalues $\lambda_1,\lambda_2,\ldots$, i.e. $P|B|P\phi_i=\lambda_i\phi$, $\phi_i\in\mathrm{Im}\,P$, we define for $f\in L^2(Y)$

$$\begin{aligned}\left(\mathrm{e}^{-tP|B|P}f\right)(t,y) &= (I-P)f(y)+\mathrm{e}^{-tP|B|P}Pf(t,y)\\ &= (I-P)f(y)+\sum_{i=1}^{\infty}\mathrm{e}^{-\lambda_i t}\langle f,\phi_i\rangle_Y\phi_i(y).\end{aligned}$$

We can also define $\mathrm{e}^{-tP|B|P}$ by

$$\mathrm{e}^{-tP|B|P}=(I-P)+\frac{1}{2\pi i}\int_C\mathrm{e}^{-tz}(z-P|B|P)^{-1}P\mathrm{d}z,$$

where C is a contour $\{r\mathrm{e}^{\mathrm{i}\pi}\mid\infty>r\geq\epsilon\}\cup\{\epsilon\mathrm{e}^{\mathrm{i}\theta}\mid\pi\geq\theta\geq-\pi\}\cup\{r\mathrm{e}^{-\mathrm{i}\pi}\mid\epsilon\leq r<\infty\}$ in $\mathbb{C}$ for small $\epsilon>0$. Then we define, for $\mathrm{Re}\,s>m=\dim Y$,

$$\zeta_{(P|B|P)}(s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\mathrm{Tr}\,\mathrm{e}^{-tP|B|P}P-l_P\right)\mathrm{d}t,\tag{2.9}$$

where $l_P=\dim(\ker B\cap\mathrm{Im}\,P)$. To show that $\zeta_{(P|B|P)}(s)$ has a regular value at $s=0$, it is enough to check that $\mathrm{Tr}\left(\mathrm{e}^{-tP|B|P}P\right)$ has an asymptotic expansion for $t\to0^+$ which has no $(\log t)^n$-terms ($n=1,2,\ldots$). We show this by comparing $\mathrm{Tr}\,\mathrm{e}^{-tP|B|P}P$ with $\mathrm{Tr}\left(\mathrm{e}^{-t\Pi_{>,\sigma^+}|B|\Pi_{>,\sigma^+}}\Pi_{>,\sigma^+}\right) = \frac{1}{2}\mathrm{Tr}\,\mathrm{e}^{-t|B|}$. Note that

$$P|B|P = \Pi_{>,\sigma^+}|B|\Pi_{>,\sigma^+} + \mathfrak{K}, \quad \mathfrak{K} = (P - \Pi_{>,\sigma^+})|B|P + \Pi_{>,\sigma^+}|B|(P - \Pi_{>,\sigma^+}),$$

where $\mathfrak{K}$ is a smoothing operator. Then simple computation leads to

$$\begin{aligned}&\mathrm{Tr}\left(\mathrm{e}^{-tP|B|P}P - \mathrm{e}^{-t\Pi_{>,\sigma^+}|B|\Pi_{>,\sigma^+}}\Pi_{>,\sigma^+}\right)\\ &\quad = \mathrm{Tr}\left\{\mathrm{e}^{-tP|B|P}(P - \Pi_{>,\sigma^+}) - t\int_0^1 \mathfrak{K}\mathrm{e}^{-t(\Pi_{>,\sigma^+}|B|\Pi_{>,\sigma^+}+u\mathfrak{K})}\Pi_{>,\sigma^+}\mathrm{d}u\right\}. \end{aligned} \tag{2.10}$$

Using the following relations:

$$G^{-1} = -G, \qquad -GPG = I - P, \qquad -G\Pi_{>,\sigma}G = I - \Pi_{>,\sigma}, \qquad G|B| = |B|G, \tag{2.11}$$

it is not difficult to check that

$$\mathrm{Tr}(P - \Pi_{>,\sigma^+}) = \mathrm{Tr}\,\mathfrak{K} = \mathrm{Tr}\left(P|B|P(P - \Pi_{>,\sigma^+}) + \mathfrak{K}\Pi_{>,\sigma^+}\right) = 0,$$

which implies that (2.10) is $o(t)$ for $t \to 0^+$. Hence $\mathrm{Tr}\left(\mathrm{e}^{-tP|B|P}P\right)$ has the same asymptotic expansion as $\mathrm{Tr}\left(\mathrm{e}^{-t\Pi_{>,\sigma^+}|B|\Pi_{>,\sigma^+}}\Pi_{>,\sigma^+}\right) = \frac{1}{2}\mathrm{Tr}\,\mathrm{e}^{-t|B|}$ at least up to order t. This implies that $\zeta_{(P|B|P)}(s)$ has a regular value at $s = 0$ and hence the zeta-determinant of $(P|B|P)$ is well defined. Similarly, if we replace $|B|$ by $\left(\sqrt{B^2+\lambda} + |B|\right)$ with $\lambda > 0$ and carry out the same computation, we have, for $t \to 0^+$,

$$\mathrm{Tr}\left(\mathrm{e}^{-t(P(\sqrt{B^2+\lambda}+|B|)P)}P - \mathrm{e}^{-t(\Pi_{>,\sigma^+}(\sqrt{B^2+\lambda}+|B|)\Pi_{>,\sigma^+})}\Pi_{>,\sigma^+}\right) = o(t),$$

which shows that $\zeta_{P(\sqrt{B^2+\lambda}+|B|)P}(s)$ has a regular value at $s = 0$ and hence the zeta-determinant of $\left(P\left(\sqrt{B^2+\lambda}+|B|\right)P\right)$ is well defined. Moreover,

$$\begin{aligned}&\log\mathrm{Det}\left(P\left(\sqrt{B^2+\lambda}+|B|\right)P\right) - \log\mathrm{Det}\left(\Pi_{>,\sigma^+}\left(\sqrt{B^2+\lambda}+|B|\right)\Pi_{>,\sigma^+}\right)\\ &\quad = \int_0^\infty \frac{1}{t}\mathrm{Tr}\left(\mathrm{e}^{-t(P(\sqrt{B^2+\lambda}+|B|)P)}P - \mathrm{e}^{-t(\Pi_{>,\sigma^+}(\sqrt{B^2+\lambda}+|B|)\Pi_{>,\sigma^+})}\Pi_{>,\sigma^+}\right)\mathrm{d}t. \end{aligned}\tag{2.12}$$

Next, we are going to use (2.12) to show that the zeta-determinant of $(I-P)(Q(\lambda)-B)(I-P)$ is well defined. We define the zeta function for $(I-P)(Q(\lambda)-B)(I-P)$ like (2.9), i.e., for $\mathrm{Re}\,s > m = \dim Y$,

$$\zeta_{((I-P)(Q(\lambda)-B)(I-P))}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}\left(\mathrm{e}^{-t(I-P)(Q(\lambda)-B)(I-P)}(I-P)\right)\mathrm{d}t.$$

Since $Q(\lambda) - \sqrt{B^2+\lambda}$ is a smoothing operator [13], we have

$$(I-P)(Q(\lambda)-B)(I-P) + P\left(\sqrt{B^2+\lambda}+|B|\right)P = \sqrt{B^2+\lambda}+|B|+\mathfrak{R}(\lambda), \tag{2.13}$$

where $\mathfrak{R}(\lambda)$ is a smoothing operator. We here note that $\left(\sqrt{B^2+\lambda}+|B|+\mathfrak{R}(\lambda)\right)$ is a self-adjoint elliptic operator on Y and hence the zeta-determinant is well defined. This fact together with (2.12) and (2.13) implies that the zeta-determinant of $(I-P)(Q(\lambda)-B)(I-P)$ is well defined and

$$\begin{aligned}\log\mathrm{Det}\left((I-P)(Q(\lambda)-B)(I-P)\right) &= \log\mathrm{Det}\left(\sqrt{B^2+\lambda}+|B|+\mathfrak{R}(\lambda)\right)\\ &\quad - \log\mathrm{Det}\left(P(\sqrt{B^2+\lambda}+|B|)P\right). \end{aligned}\tag{2.14}$$

Then Eq. (2.8) yields the following result.

Theorem 2.2. *For some real numbers $a_0, a_1, \ldots, a_{\nu-1}$, the following equality holds:*

$$\log \operatorname{Det}\left(D^2_{M,P}+\lambda\right) - \log \operatorname{Det}\left(D^2_{M,\gamma_0}+\lambda\right) = \sum_{j=0}^{\nu-1} a_j + \log \operatorname{Det}\left((I-P)\left(Q(\lambda)-B\right)(I-P)\right) + \log \operatorname{Det}^*\left(P|B|P\right),$$

where $((I-P)(Q(\lambda)-B)(I-P))$ and $P|B|P$ are considered as operators defined on $\operatorname{Im}(I-P)$ *and* $\operatorname{Im} P$*, respectively.*

We next discuss the constant a_0 in the above theorem. It was shown in the Appendix of [5] that for $\lambda \to \infty$, $\log \operatorname{Det}\left(\sqrt{B^2+\lambda}+|B|+\Re(\lambda)\right)$ and $\log \operatorname{Det}\left(\sqrt{B^2+\lambda}+|B|\right)$ have asymptotic expansions, which are exactly the same. Direct computation shows that $\log \operatorname{Det}\left(D^2_{M,P}+\lambda\right)$ and $\log \operatorname{Det}\left(D^2_{M,\gamma_0}+\lambda\right)$ have asymptotic expansions, whose zero-coefficients are zeros (cf. [27] or Proposition 2.7 in [12]). Hence, (2.12) and (2.14) show that the zero-coefficient in the asymptotic expansion of $\log \operatorname{Det}\left((I-P)(Q(\lambda)-B)(I-P)\right)$ is the same as that of $\frac{1}{2}\log \operatorname{Det}\left(\sqrt{B^2+\lambda}+|B|\right)$ for $\lambda \to \infty$. This implies that $-a_0$ is the sum of $\log \operatorname{Det}^*(P|B|P)$ and the zero-coefficient in the asymptotic expansion of $\frac{1}{2}\log \operatorname{Det}\left(\sqrt{B^2+\lambda}+|B|\right)$, which can be computed as follows.

Lemma 2.3. *For $1 \le k \in \mathbb{Z}$ let us define $f_k(s,\lambda)$ by*

$$f_k(s,\lambda) = \frac{1}{\Gamma(s)}\int_0^\infty t^{\frac{s+k}{2}-1} \operatorname{Tr}\left(|B|^k e^{-t(B^2+\lambda)}\right) dt.$$

Then the zero-coefficients in the asymptotic expansions of $f_k(0,\lambda)$ and $-f_k'(0,\lambda)$, for $\lambda \to \infty$, are zeros.

Proof. We first note that $\operatorname{Tr}\left(|B|^k e^{-tB^2}\right)$ has the following asymptotic expansion for $t \to 0^+$ (Theorem 2.7 in [10] or [4]):

$$\operatorname{Tr}\left(|B|^k e^{-tB^2}\right) \sim \sum_{j=0}^\infty b_j^{(k)} t^{\frac{j-m-k}{2}} + \sum_{j=0}^\infty \left(c_j^{(k)} \log t + d_j^{(k)}\right) t^j. \tag{2.15}$$

Then direct computation shows that the zero-coefficients of $f_k(0,\lambda)$ and $-\zeta_k'(0,\lambda)$ for $\lambda \to \infty$ are $2b_m^{(k)}$ and $\Gamma'(1)b_m^{(k)}$. On the other hand, we note that

$$\zeta_{|B|}(s) = \frac{1}{\Gamma(\frac{s+k}{2})}\int_0^\infty t^{\frac{s+k}{2}-1} \operatorname{Tr}\left(|B|^k e^{-tB^2}\right) dt. \tag{2.16}$$

Then (2.15) shows that the RHS of (2.16) has a pole at $s=0$ with residue $\frac{2b_m^{(k)}}{\Gamma(\frac{k}{2})}$. Since $\zeta_{|B|}(s)$ has a regular value at $s=0$, this fact implies that each $b_m^{(k)}=0$ for $k \ge 1$, which completes the proof of the lemma. □

Lemma 2.4. *The zero-coefficient in the asymptotic expansion of* $\log \operatorname{Det}\left(\sqrt{B^2+\lambda}+|B|\right)$ *for $\lambda \to \infty$ is zero.*

Proof. We note that

$$\begin{aligned}
\zeta_{(\sqrt{B^2+\lambda}+|B|)}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} \operatorname{Tr} e^{-t(\sqrt{B^2+\lambda}+|B|)} dt \\
&= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} \sum_{q=0}^\infty \frac{t^q}{q!} \operatorname{Tr}\left(\left(\sqrt{B^2+\lambda}-|B|\right)^q e^{-2t\sqrt{B^2+\lambda}}\right) dt \\
&= \sum_{q=0}^\infty \frac{1}{q!}\frac{1}{\Gamma(s)}\int_0^\infty t^{s+q-1} \operatorname{Tr}\left(\left(\sqrt{B^2+\lambda}-|B|\right)^q e^{-2t\sqrt{B^2+\lambda}}\right) dt.
\end{aligned}$$

We set

$$\zeta_q(s,\lambda) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s+q-1} \mathrm{Tr}\left(\left(\sqrt{B^2+\lambda} - |B| \right)^q \mathrm{e}^{-2t\sqrt{B^2+\lambda}} \right) \mathrm{d}t.$$

In the case of $q = 0$, the zero-coefficient in the asymptotic expansion of $-\zeta_0'(0,\lambda)$ for $\lambda \to \infty$ is $\log 2 \cdot \left(\zeta_{B^2}(0) + \dim\ker B\right)$. For $q \geq 1$,

$$\begin{aligned} \zeta_q(s,\lambda) &= \sum_{k=0}^{q} (-1)^k \binom{q}{k} \frac{1}{\Gamma(s)} \int_0^\infty t^{s+q-1} \mathrm{Tr}\left(\left(\sqrt{B^2+\lambda} \right)^{q-k} |B|^k \mathrm{e}^{-2t\sqrt{B^2+\lambda}} \right) \mathrm{d}t \\ &= 2^{-s-q} \sum_{k=0}^{q} (-1)^k \binom{q}{k} \frac{\Gamma(s+q)}{\Gamma(\frac{s+k}{2})} \frac{1}{\Gamma(s)} \int_0^\infty t^{\frac{s+k}{2}-1} \mathrm{Tr}\left(|B|^k \mathrm{e}^{-t(B^2+\lambda)} \right) \mathrm{d}t. \end{aligned}$$

Lemma 2.3 shows that the zero-coefficient of $-\zeta_q'(0,\lambda)$ is obtained only for $k = 0$, which is $-\frac{1}{q\cdot 2^q} \cdot \left(\zeta_{B^2}(0) + \dim\ker B\right)$. Since $\log 2 = \sum_{q=1}^{\infty} \frac{1}{q\cdot 2^q}$, this completes the proof of the lemma. □

Lemma 2.4 shows that

$$a_0 + \log \mathrm{Det}^* \left(P|B|P\right) = 0$$

and hence we have

$$\log \mathrm{Det}\left(D^2_{M,P} + \lambda\right) - \log \mathrm{Det}\left(D^2_{M,\gamma_0} + \lambda\right) = \sum_{j=1}^{\nu-1} a_j \lambda^j + \log \mathrm{Det}\left((I-P)\left(Q(\lambda)-B\right)(I-P)\right). \tag{2.17}$$

Finally, we are going to discuss the behavior of (2.17) as $\lambda \to 0$. We define $q = \dim\ker D^2_{M,P}$. Since D^2_{M,γ_0} is an invertible operator, we have

$$\begin{aligned} \log \mathrm{Det}\left(D^2_{M,\gamma_0} + \lambda\right) &= \log \mathrm{Det}\, D^2_{M,\gamma_0} + o(\lambda), \\ \log \mathrm{Det}\left(D^2_{M,P} + \lambda\right) &= q \cdot \log\lambda + \log \mathrm{Det}^*\, D^2_{M,P} + o(\lambda). \end{aligned} \tag{2.18}$$

The following lemma shows the relation between $\ker D^2_{M,P}$ and $\ker\left((I-P)(Q-B)(I-P)\right)$.

Lemma 2.5. (1) $\ker(Q-B) = \{\phi|_Y \mid D_M\phi = 0\} = \mathrm{Im}\,\mathfrak{C}$, *and hence* $(Q-B)$ *maps* $\mathrm{Im}\,(I-\mathfrak{C})$ *onto* $\mathrm{Im}\,(I-\mathfrak{C})$.
(2) $\ker\left((I-P)(Q-B)(I-P)\right) = \ker(Q-B) \cap \mathrm{Im}(I-P) = \left\{\phi|_Y \mid \phi \in \ker D_{M,P}\right\}$, *and* $\dim\ker\left((I-P)(Q-B)(I-P)\right) = \dim\ker D_{M,P}$.

Proof. The second assertion follows from the first assertion and the unique continuation property of D_M. If $\phi \in C^\infty(M)$ satisfies $D_M\phi = 0$, $Q(\phi|_Y) = -(\partial_u\phi)|_Y = B(\phi|_Y)$, and hence $\phi|_Y \in \ker(Q-B)$. Conversely, suppose that $f \in \ker(Q-B)$. We choose the unique section $\phi \in C^\infty(M)$ so that

$$D^2_M\phi = 0, \qquad \phi|_Y = f.$$

By the Green Theorem (cf. Lemma 3.1 in [7]),

$$\begin{aligned} 0 &= \langle D^2_M\phi, \phi\rangle_M = \langle D_M\phi, D_M\phi\rangle_M + \langle (D_M\phi)|_Y, G\phi|_Y\rangle_Y \\ &= \langle D_M\phi, D_M\phi\rangle_M + \langle -Q(f) + Bf, f\rangle_Y = \langle D_M\phi, D_M\phi\rangle_M, \end{aligned}$$

which implies that $D_M\phi = 0$ and hence $f \in \mathrm{Im}\,\mathfrak{C}$. Since $(Q-B)$ is self-adjoint, it maps $\mathrm{Im}\,(I-\mathfrak{C})$ onto itself. □

Now let us denote the eigenvalues of $(I-P)(Q(\lambda)-B)(I-P)$ on $\mathrm{Im}(I-P)$ by

$$0 < \kappa_1(\lambda) \leq \cdots \leq \kappa_q(\lambda) < \kappa_{q+1}(\lambda) \leq \cdots$$

and the corresponding orthonormal eigensections by

$$h_1(\lambda), \cdots, h_q(\lambda), h_{q+1}(\lambda), \cdots.$$

Then for $1 \leq j \leq q$,

$$\lim_{\lambda \to 0} \kappa_j(\lambda) = 0, \qquad \lim_{\lambda \to 0} h_j(\lambda) = h_j,$$

where $\{h_1, h_2, \ldots, h_q\}$ is an orthonormal basis of $\ker((I-P)(Q-B)(I-P))$. This leads to

$$\log \mathrm{Det}((I-P)(Q(\lambda)-B)(I-P)) = \log \kappa_1(\lambda) \cdots \kappa_q(\lambda) + \log \mathrm{Det}^*((I-P)(Q-B)(I-P)) + o(\lambda). \tag{2.19}$$

The second assertion in Lemma 2.5 shows that each h_j can be extended to a global section $\psi_j \in C^\infty(M)$ such that

$$D_{M,P}\psi_j = 0, \qquad \psi_j|_Y = h_j. \tag{2.20}$$

The next result shows the behavior of $\kappa_j(\lambda)$ for $\lambda \to 0^+$.

Lemma 2.6.

$$\lim_{\lambda \to 0} \frac{\kappa_j(\lambda)}{\lambda} = \langle \psi_j, \psi_j \rangle_M, \quad \textit{and} \quad \langle \psi_i, \psi_j \rangle_M = 0 \quad \textit{for } i \neq j,\ 1 \leq i, j \leq q,$$

and hence

$$\log \kappa_1(\lambda) \cdots \kappa_q(\lambda) = q \log \lambda + \log \det \left(\langle \psi_i, \psi_j \rangle_M \right) + o(\lambda).$$

Proof. Since $(I-P)(h_k(\lambda)) = h_k(\lambda)$ and $(I-P)(h_k) = h_k$ for $1 \leq k \leq q$, we have

$$\begin{aligned} \kappa_j(\lambda)\langle h_j(\lambda), h_k \rangle_Y &= \langle ((I-P)(Q(\lambda)-B)(I-P))\, h_j(\lambda), h_k \rangle_Y \\ &= \langle (Q(\lambda)-B)\, h_j(\lambda), h_k \rangle_Y. \end{aligned} \tag{2.21}$$

Let $\psi_j(\lambda)$ be the smooth section on M such that

$$(D_M^2 + \lambda)\psi_j(\lambda) = 0, \qquad \psi_j(\lambda)|_Y = h_j(\lambda).$$

Using the Green formula and (2.20), we have

$$\begin{aligned} 0 &= \left\langle \left(D_M^2 + \lambda \right) (\psi_j(\lambda)), \psi_k \right\rangle_M = \lambda \langle \psi_j(\lambda), \psi_k \rangle_M + \langle D_M^2 \psi_j(\lambda), \psi_k \rangle_M \\ &= \lambda \langle \psi_j(\lambda), \psi_k \rangle_M + \langle D_M \psi_j(\lambda), D_M \psi_k \rangle_M + \int_Y \left(D_M \psi_j(\lambda)|_Y, G\psi_k|_Y \right) dvol(Y) \\ &= \lambda \langle \psi_j(\lambda), \psi_k \rangle_M + \langle \left((\partial_u + B)\psi_j(\lambda) \right)|_Y, h_k \rangle_Y \end{aligned}$$

and hence

$$\langle (Q(\lambda)-B)\, h_j(\lambda), h_k \rangle_Y = \lambda \langle \psi_j(\lambda), \psi_k \rangle_M. \tag{2.22}$$

Eqs. (2.21) and (2.22) show that

$$\kappa_j(\lambda)\langle h_j(\lambda), h_k \rangle_Y = \lambda \langle \psi_j(\lambda), \psi_k \rangle_M. \tag{2.23}$$

Since $\lim_{\lambda \to 0} \psi_j(\lambda)|_Y = \psi_j|_Y$, the unique continuation property of D_M implies $\lim_{\lambda \to 0} \psi_j(\lambda) = \psi_j$. Since $\langle h_j, h_k \rangle_Y = \delta_{jk}$, the result follows. □

Lemma 2.6 with (2.17)–(2.19) imply Theorem 1.1.

3. The proof of Theorems 1.2 and 1.3

In this section we are going to prove Theorems 1.2 and 1.3. Note that $\mathrm{Im}\,\mathfrak{C} = \mathrm{graph}(K)$ and $\mathrm{Im}\,(I-\mathfrak{C}) = \mathrm{graph}(-K)$. Since $(I-K)$ is a map from $L^2(Y, E_Y^+)$ onto $\mathrm{Im}\,(I-\mathfrak{C})$, Lemma 2.5 shows that $(I-\mathfrak{C})(Q-B)(I-\mathfrak{C})$ has the same spectrum as $(I-K)^{-1}(Q-B)(I-K)$ and hence

$$\log \mathrm{Det}((I-\mathfrak{C})(Q-B)(I-\mathfrak{C})) = \log \mathrm{Det}\left((I-K)^{-1} (Q-B)(I-K) \right). \tag{3.1}$$

We note again $\mathrm{Im}(I-P) = \mathrm{graph}(-T)$ and define U, L by

$$U = \mathrm{Im}\,(I-P) \cap \mathrm{Im}\,\mathfrak{C} = \ker(I-P)(Q-B)(I-P) = \{\phi|_Y \mid D_{M,P}\phi = 0\},$$
$$L = (I-T)^{-1}\,(U) = (I+K)^{-1}\,(U) = \{x \in L^2(E_Y^+) \mid Tx = -Kx\}. \tag{3.2}$$

We also denote by $\mathrm{Im}\,(I-P)^*$ and $L^2(E_Y^+)^*$ the orthogonal complements of U, L so that

$$\mathrm{Im}\,(I-P) = \mathrm{Im}\,(I-P)^* \oplus U, \qquad L^2(E_Y^+) = L^2(E_Y^+)^* \oplus L.$$

Then it is not difficult to see that $\ker(I + K^{-1}T) = L$ and

$$(I + K^{-1}T)|_{L^2(E_Y^+)^*} : L^2(E_Y^+)^* \to L^2(E_Y^+)^* \tag{3.3}$$

is invertible.

Using the first assertion of Lemma 2.5 and the following identity:

$$(I-K) = \frac{1}{2}(I+T)(I-T^{-1}K) + \frac{1}{2}(I-T)(I+T^{-1}K),$$
$$(I-T) = \frac{1}{2}(I+K)(I-K^{-1}T) + \frac{1}{2}(I-K)(I+K^{-1}T), \tag{3.4}$$

we have

$$\begin{aligned}
&\log \mathrm{Det}^*\,((I-P)(Q-B)(I-P)) = \log \mathrm{Det}\left((I-P)(Q-B)(I-P) + \mathrm{pr}_U\right)\\
&\quad = \log \mathrm{Det}\left((I-T)^{-1}(I-P)(Q-B)(I-P)(I-T) + \mathrm{pr}_L\right)\\
&\quad = \log \mathrm{Det}\left((I-T)^{-1}(I-P)(I-K)(I-K)^{-1}(Q-B)(I-T) + \mathrm{pr}_L\right)\\
&\quad = \log \mathrm{Det}\left(\frac{1}{2}(I+T^{-1}K)(I-K)^{-1}(Q-B)(I-K)\frac{1}{2}(I+K^{-1}T) + \mathrm{pr}_L\right)\\
&\quad = \log \mathrm{Det}\left(\frac{1}{4}(I+K^{-1}T)(I+T^{-1}K)(I-K)^{-1}(Q-B)(I-K) + \mathrm{pr}_L\right)\\
&\quad = \log \mathrm{Det}\left(\frac{1}{4}(I+K^{-1}T)(I+T^{-1}K) + \mathrm{pr}_L(I-K)^{-1}(Q-B)^{-1}(I-K)\right)\\
&\qquad \times ((I-K)^{-1}(Q-B)(I-K))\\
&\quad = \log \mathrm{det}_{\mathrm{Fr}}\left(\frac{1}{4}(I+K^{-1}T)(I+T^{-1}K) + \mathrm{pr}_L(I-K)^{-1}(Q-B)^{-1}(I-K)\mathrm{pr}_L\right)\\
&\qquad + \log \mathrm{Det}\left((I-K)^{-1}(Q-B)(I-K)\right)\\
&\quad = \log\left|\mathrm{det}^*_{\mathrm{Fr}}\frac{1}{2}(I+T^{-1}K)\right|^2 + \log\det\left(\mathrm{pr}_L(I-K)^{-1}(Q-B)^{-1}(I-K)\mathrm{pr}_L\right)\\
&\qquad + \log \mathrm{Det}\,((I-\mathfrak{C})(Q-B)(I-\mathfrak{C}))\,.
\end{aligned} \tag{3.5}$$

Lemma 3.1.

$$\det\left(\mathrm{pr}_L(I-K)^{-1}(Q-B)^{-1}(I-K)\mathrm{pr}_L\right) = \det V_{M,P},$$

where $V_{M,P}$ is a $q \times q$ matrix defined in (1.5).

Proof. Since $(I-K) : L \to GU = \mathrm{Im}(I-\mathfrak{C}) \cap \mathrm{Im}\,P$ is an isomorphism (cf. (3.2)), we have

$$\det\left(\mathrm{pr}_L(I-K)^{-1}(Q-B)^{-1}(I-K)\mathrm{pr}_L\right) = \det\left(\mathrm{pr}_{GU}(Q-B)^{-1}\mathrm{pr}_{GU}\right).$$

Let $\{h_1, \ldots, h_q\}$ be an orthonormal basis for U. Then $\{Gh_1, \ldots, Gh_q\}$ is an orthonormal basis for GU. Suppose that $(Q-B)^{-1}Gh_i = f_i$ and choose ϕ_i such that $D_M^2\phi_i = 0$ and $\phi_i|_Y = f_i$. Using the Green formula, we have

$$0 = \langle D_M^2\phi_i, \phi_j\rangle_M = \langle D_M\phi_i, D_M\phi_j\rangle_M + \langle D_M\phi_i|_Y, G\phi_j|_Y\rangle_Y$$
$$= \langle D_M\phi_i, D_M\phi_j\rangle_M + \langle(\partial_u + B)\phi_i|_Y, f_j\rangle_Y = \langle D_M\phi_i, D_M\phi_j\rangle_M + \langle(-Q + B) f_i, f_j\rangle_Y,$$

which shows that

$$\langle(Q - B)^{-1}Gh_i, Gh_j\rangle_Y = \langle f_i, (Q - B) f_j\rangle_Y = \langle(Q - B) f_i, f_j\rangle_Y = \langle D_M\phi_i, D_M\phi_j\rangle_M.$$

We note that

$$D_M (D_M\phi_i) = 0, \qquad (D_M\phi_i)|_Y = G(\partial_u + B)\phi_i|_Y = G(-Q + B) f_i = -GGh_i = h_i,$$

which completes the proof of the lemma. □

Theorem 1.2 follows from Theorem 1.1, (3.5) and Lemma 3.1.

Next, we are going to prove Theorem 1.3 by using a similar method. Theorem 1.1 and (1.11) lead to the following equality:

$$\log \mathrm{Det}^* \widetilde{D}^2 - \log \mathrm{Det}\, D^2_{M_1,\mathfrak{C}_1} - \log \mathrm{Det}\, D^2_{M_2,\mathfrak{C}_2} = -\log 2 \cdot (\zeta_{B^2}(0) + l) + \log\det A_0$$
$$+ \log \mathrm{Det}^* (Q_1 + Q_2) - \log \mathrm{Det}((I - \mathfrak{C}_1)(Q_1 + B)$$
$$\times (I - \mathfrak{C}_1)) - \log \mathrm{Det}((I - \mathfrak{C}_2)(Q_2 - B)(I - \mathfrak{C}_2)). \tag{3.6}$$

The following lemma can be checked in the same way as Lemma 2.5.

Lemma 3.2.

$$\ker (Q_1 + B) = \{\phi|_Y \mid D_{M_1}\phi = 0\} = \mathrm{Im}\,\mathfrak{C}_1, \qquad \ker (Q_2 - B) = \{\psi|_Y \mid D_{M_2}\psi = 0\} = \mathrm{Im}\,\mathfrak{C}_2,$$
$$\ker(Q_1 + Q_2) = \mathrm{Im}\,\mathfrak{C}_1 \cap \mathrm{Im}\,\mathfrak{C}_2 = \{\tilde{\phi}|_Y \mid \widetilde{D}\tilde{\phi} = 0\}.$$

Lemma 3.2 implies that

$$C^\infty(E|_Y) = \ker(Q_1 + Q_2) \oplus (\mathrm{Im}\,(I - \mathfrak{C}_1) + \mathrm{Im}\,(I - \mathfrak{C}_2)),$$

where $\dim(\mathrm{Im}\,(I - \mathfrak{C}_1) \cap \mathrm{Im}\,(I - \mathfrak{C}_2)) = \dim\ker (Q_1 + Q_2) = \dim\ker \widetilde{D} = q$. Using Lemma 3.2 and (3.4) with $K = K_1$ and $T = K_2$, we have for $x \in C^\infty(Y, E_Y^+)$

$$(Q_1 + Q_2)(I - K_1)x = (Q_1 + B)(I - K_1)x + (Q_2 - B)(I - K_1)x$$
$$= (I - \mathfrak{C}_1)(Q_1 + B)(I - \mathfrak{C}_1)(I - K_1)x$$
$$+ (I - \mathfrak{C}_2)(Q_2 - B)(I - \mathfrak{C}_2)(I - K_2)\frac{I + K_2^{-1}K_1}{2}x.$$

Similarly,

$$(Q_1 + Q_2)(I - K_2)y = (I - \mathfrak{C}_1)(Q_1 + B)(I - \mathfrak{C}_1)(I - K_1)\frac{I + K_1^{-1}K_2}{2}y$$
$$+ (I - \mathfrak{C}_2)(Q_2 - B)(I - \mathfrak{C}_2)(I - K_2)y.$$

Recall that $\ker (Q_1 + Q_2) = \{(I + K_1)x \mid K_1x = K_2x\}$ and denote it by H. We now define subspaces $\widetilde{H}_\pm$ of $\mathrm{Im}\,(I - \mathfrak{C}_1) \oplus \mathrm{Im}\,(I - \mathfrak{C}_2)$ by

$$\widetilde{H}_+ = \{(I - K_1)x, (I - K_2)x \mid K_1x = K_2x\}, \qquad \widetilde{H}_- = \{(I - K_1)x, -(I - K_2)x \mid K_1x = K_2x\},$$

and consider the following diagram:

$$\begin{array}{ccc} \mathrm{Im}\,(I - \mathfrak{C}_1) \oplus \mathrm{Im}\,(I - \mathfrak{C}_2) & \xrightarrow{\widetilde{R}} & \mathrm{Im}\,(I - \mathfrak{C}_1) \oplus \mathrm{Im}\,(I - \mathfrak{C}_2) \\ \Phi\downarrow & & \downarrow\Phi \\ (\mathrm{Im}\,(I - \mathfrak{C}_1) + \mathrm{Im}\,(I - \mathfrak{C}_2)) \oplus \widetilde{H}_- & \xrightarrow{\widetilde{Q}} & (\mathrm{Im}\,(I - \mathfrak{C}_1) + \mathrm{Im}\,(I - \mathfrak{C}_2)) \oplus \widetilde{H}_-, \end{array} \tag{3.7}$$

where Φ, $\widetilde{Q}$ and $\widetilde{R}$ are defined as follows:

$$\Phi\left((I-K_1)x,(I-K_2)y\right)=\Big((I-K_1)x+(I-K_2)y,\operatorname{pr}_{\widetilde{H}_-}\left((I-K_1)x,(I-K_2)y\right)\Big),$$

$$\widetilde{Q}\,(a,b)=\Big(\left(Q_1+Q_2\right)(a)\,,\operatorname{pr}_{\widetilde{H}_-}\widetilde{R}\,\Phi^{-1}(a,b)\Big),$$

$$\begin{aligned}\widetilde{R}&=\begin{pmatrix}\mathfrak{S}_1 & \mathfrak{S}_1(I-K_1)\dfrac{I+K_1^{-1}K_2}{2}(I-K_2)^{-1}\\ \mathfrak{S}_2(I-K_2)\dfrac{I+K_2^{-1}K_1}{2}(I-K_1)^{-1} & \mathfrak{S}_2\end{pmatrix}\operatorname{pr}_{(\widetilde{H}_-)^\perp}+r_0\operatorname{pr}_{\widetilde{H}_-}\\ &=\begin{pmatrix}\mathfrak{S}_1 & 0\\ 0 & \mathfrak{S}_2\end{pmatrix}\begin{pmatrix}I & (I-K_1)\dfrac{I+K_1^{-1}K_2}{2}(I-K_2)^{-1}\\ (I-K_2)\dfrac{I+K_2^{-1}K_1}{2}(I-K_1)^{-1} & I\end{pmatrix}+r_0\operatorname{pr}_{\widetilde{H}_-},\end{aligned}$$

where $\mathfrak{S}_1=(I-\mathfrak{C}_1)\,(Q_1+B)\,(I-\mathfrak{C}_1)$, $\mathfrak{S}_2=(I-\mathfrak{C}_2)\,(Q_2-B)\,(I-\mathfrak{C}_2)$ and r_0 is a positive real number such that $r_0\notin\operatorname{Spec}(Q_1+Q_2)$. Then all maps are invertible and the diagram (3.7) commutes. Hence,

$$\begin{aligned}\log\operatorname{Det}\widetilde{Q}&=q\log r_0+\log\operatorname{Det}^*\left(Q_1+Q_2\right)=\log\operatorname{Det}\left(I-\mathfrak{C}_1\right)\left(Q_1+B\right)\left(I-\mathfrak{C}_1\right)\\ &\quad+\log\operatorname{Det}\left(I-\mathfrak{C}_2\right)\left(Q_2-B\right)\left(I-\mathfrak{C}_2\right)+\log\det{}_{\mathrm{Fr}}\left(\alpha+\beta\right),\end{aligned}\tag{3.8}$$

where

$$\alpha=\begin{pmatrix}I & (I-K_1)\dfrac{I+K_1^{-1}K_2}{2}(I-K_2)^{-1}\\ (I-K_2)\dfrac{I+K_2^{-1}K_1}{2}(I-K_1)^{-1} & I\end{pmatrix},$$

$$\beta=r_0\begin{pmatrix}{\mathfrak{S}_1}^{-1} & 0\\ 0 & {\mathfrak{S}_2}^{-1}\end{pmatrix}\operatorname{pr}_{\widetilde{H}_-}.$$

We note that $H=\operatorname{Im}\mathfrak{C}_1\cap\operatorname{Im}\mathfrak{C}_2$ implies $GH=\operatorname{Im}\left(I-\mathfrak{C}_1\right)\cap\operatorname{Im}\left(I-\mathfrak{C}_2\right)$ and hence

$$\operatorname{Im}\left(I-\mathfrak{C}_1\right)\oplus\operatorname{Im}\left(I-\mathfrak{C}_2\right)=\left(\left(\operatorname{Im}\left(I-\mathfrak{C}_1\right)\ominus GH\right)\oplus\left(\operatorname{Im}\left(I-\mathfrak{C}_2\right)\ominus GH\right)\right)\oplus\widetilde{H}_+\oplus\widetilde{H}_-.$$

Since α maps $(\widetilde{H}_-)^\perp$ onto $(\widetilde{H}_-)^\perp$ and $\alpha|_{\widetilde{H}_+}=2\mathrm{Id}|_{\widetilde{H}_+}$,

$$\begin{aligned}\log\det{}_{\mathrm{Fr}}\left(\alpha+\beta\right)&=q\log 2+\log\det{}_{\mathrm{Fr}}\Big(\alpha|_{\oplus_{i=1}^{2}(\operatorname{Im}(I-\mathfrak{C}_i)\ominus GH)}\Big)+q\log r_0+\log\det\Big(\operatorname{pr}_{\widetilde{H}_-}\beta\Big)\\ &=q\log 2+q\log r_0+\log\left|\det{}^*_{\mathrm{Fr}}\left(\frac{1}{2}\left(I-K_1^{-1}K_2\right)\right)\right|^2\\ &\quad+\log\det\left(\operatorname{pr}_{\widetilde{H}_-}\begin{pmatrix}\mathfrak{S}_1^{-1} & 0\\ 0 & \mathfrak{S}_2^{-1}\end{pmatrix}\operatorname{pr}_{\widetilde{H}_-}\right),\end{aligned}\tag{3.9}$$

where $q=\dim\widetilde{H}_+=\dim\ker\widetilde{D}$. Let $\{h_1,\ldots,h_q\}$ be an orthonormal basis of $\operatorname{Im}\mathfrak{C}_1\cap\operatorname{Im}\mathfrak{C}_2$. Then $\{Gh_1,\ldots,Gh_q\}$ is an orthonormal basis for $\operatorname{Im}\left(I-\mathfrak{C}_1\right)\cap\operatorname{Im}\left(I-\mathfrak{C}_2\right)$ and this gives an orthonormal basis $\left\{\frac{1}{\sqrt{2}}(Gh_1,-Gh_1),\ldots,\frac{1}{\sqrt{2}}(Gh_q,-Gh_q)\right\}$ for $\widetilde{H}_-$. We note that

$$\left\langle\operatorname{pr}_{\widetilde{H}_-}\begin{pmatrix}\mathfrak{S}_1^{-1} & 0\\ 0 & \mathfrak{S}_2^{-1}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}Gh_i\\ -Gh_i\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix}Gh_j\\ -Gh_j\end{pmatrix}\right\rangle=\frac{1}{2}\left(\langle\mathfrak{S}_1^{-1}Gh_i,Gh_j\rangle+\langle\mathfrak{S}_2^{-1}Gh_i,Gh_j\rangle\right),$$

which shows that

$$\log\det\left(\operatorname{pr}_{\widetilde{H}_-}\begin{pmatrix}\mathfrak{S}_1^{-1} & 0\\ 0 & \mathfrak{S}_2^{-1}\end{pmatrix}\operatorname{pr}_{\widetilde{H}_-}\right)=-q\log 2+\log\det\left(\operatorname{pr}_{GH}\left(\mathfrak{S}_1^{-1}+\mathfrak{S}_2^{-1}\right)\operatorname{pr}_{GH}\right).\tag{3.10}$$

Lemma 3.3.

$$\det\left(\mathrm{pr}_{GH}\left((Q_1+B)^{-1}+(Q_2-B)^{-1}\right)\mathrm{pr}_{GH}\right)=\det A_0,$$

where A_0 is a $q\times q$ matrix defined in (1.10).

Proof. Suppose that $\{Gh_1,\ldots,Gh_q\}$ is an orthonormal basis for $\mathrm{Im}\,(I-\mathfrak{C}_1)\cap\mathrm{Im}\,(I-\mathfrak{C}_2)$ and define $(Q_1+B)^{-1}Gh_i=f_i$, $(Q_2-B)^{-1}Gh_j=g_j$. We choose $\phi_1,\ldots,\phi_q\in C^\infty(M_1)$, $\psi_1,\ldots,\psi_q\in C^\infty(M_2)$ such that

$$D^2_{M_1}\phi_i=0,\qquad D^2_{M_2}\psi_i=0,\qquad \phi_i|_Y=f_i,\qquad \psi_j|_Y=g_j.$$

Using the Green formula,

$$\begin{aligned}0=\langle D^2_{M_1}\phi_i,\phi_j\rangle_{M_1}&=\langle D_{M_1}\phi_i,D_{M_1}\phi_j\rangle_{M_1}-\langle(D_{M_1}\phi_i)|_Y,(G\phi_j)|_Y\rangle_Y\\&=\langle D_{M_1}\phi_i,D_{M_1}\phi_j\rangle_{M_1}-\langle((\partial_u+B)\phi_i)|_Y,f_j\rangle_Y\\&=\langle D_{M_1}\phi_i,D_{M_1}\phi_j\rangle_{M_1}-\langle(Q_1+B)f_i,f_j\rangle_Y.\end{aligned}$$

In the same way as in Lemma 3.1,

$$\begin{aligned}\langle(Q_1+B)^{-1}Gh_i,Gh_j\rangle_Y&=\langle f_i,(Q_1+B)f_j\rangle_Y=\langle(Q_1+B)f_i,f_j\rangle_Y\\&=\langle D_{M_1}\phi_i,D_{M_1}\phi_j\rangle_{M_1}=\langle -D_{M_1}\phi_i,-D_{M_1}\phi_j\rangle_{M_1}.\end{aligned}\tag{3.11}$$

Note that

$$D_{M_1}(-D_{M_1}\phi_i)=0,\qquad(-D_{M_1}\phi_i)|_Y=-G(\partial_u+B)\phi_i|_Y=-G(Q_1+B)f_i=-GGh_i=h_i.\tag{3.12}$$

In the same way,

$$\langle(Q_2-B)^{-1}Gh_i,Gh_j\rangle_Y=\langle D_{M_2}\psi_i,D_{M_2}\psi_j\rangle_{M_2},\tag{3.13}$$

where

$$D_{M_2}(D_{M_2}\psi_i)=0,\qquad(D_{M_2}\psi_i)|_Y=G(\partial_u+B)\psi_i|_Y=G(-Q_2+B)g_i=-GGh_i=h_i.\tag{3.14}$$

Setting

$$\Phi_i=(-D_{M_1}\phi_i)\cup_Y(D_{M_2}\psi_i),$$

Lemma 3.2 with (3.12) and (3.14) shows that Φ_i is a smooth section and belongs to $\ker\widetilde{D}$. Hence, (3.11) and (3.13) show that

$$\langle(Q_1+B)^{-1}Gh_i,Gh_j\rangle_Y+\langle(Q_2-B)^{-1}Gh_i,Gh_j\rangle_Y=\langle\Phi_i,\Phi_j\rangle_{\widetilde{M}},$$

which completes the proof of the lemma. □

The first assertion in Theorem 1.3 is obtained by the above lemma with (3.6) and (3.8)–(3.10). Theorem 1.2 together with the first assertion in Theorem 1.3 yields the second assertion in Theorem 1.3.

4. The proof of Theorem 1.5

In this section we are going to prove Theorem 1.5. To prove the first assertion we begin with the following fact:

$$\ker(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}}=\{f\in C^\infty(Y)\mid f\in\mathrm{Im}\,\tau^-\cap\mathrm{Im}\,\sigma^+\}.\tag{4.1}$$

By Corollary 1.4 we have

$$\begin{aligned}&\log\mathrm{Det}^*(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}}-\log\mathrm{Det}(-\partial_u^2+B^2)_{N_{0,r},\gamma_0,\gamma_r}\\&\quad=\log\mathrm{Det}^*(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}}-\log\mathrm{Det}(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}\\&\qquad+\log\mathrm{Det}(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}-\log\mathrm{Det}(-\partial_u^2+B^2)_{N_{0,r},\gamma_0,\gamma_r}\\&\quad=\log\mathrm{Det}^*(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}}-\log\mathrm{Det}(-\partial_u^2+B^2)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}\\&\qquad-\frac{l}{2}\cdot\log r+\frac12\log 2\cdot\zeta_{B^2}(0)+\frac14\log\mathrm{Det}^*B^2+\frac12\log\det{}_{\mathrm{Fr}}\left(I+\frac{e^{-r|B|}}{e^{r|B|}-e^{-r|B|}}\mathrm{pr}_{(\ker B)^\perp}\right).\end{aligned}\tag{4.2}$$

To establish a formula analogous to Corollary 1.4 for $\log \mathrm{Det}^*(-\partial_u^2 + B^2)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} - \log \mathrm{Det}(-\partial_u^2 + B^2)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}$, we consider, as in the proof of Theorem 1.1,

$$\log \mathrm{Det}\left(-\partial_u^2 + B^2 + \lambda\right)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} - \log \mathrm{Det}\left(-\partial_u^2 + B^2 + \lambda\right)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}.$$

To define the operator $R_{\mathrm{cyl}}(\lambda) : C^\infty(Y_r) \to C^\infty(Y_r)$ corresponding to $R_P(\lambda)$ in Theorem 1.1, we introduce the Poisson operator $P_{\mathrm{cyl}}(\lambda) : C^\infty(Y_r) \to C^\infty(N_{0,r})$ associated with the boundary condition $\Pi_{>,\tau^+}$ on Y_0, which is characterized as follows:

$$(-\partial_u^2 + B^2 + \lambda)P_{\mathrm{cyl}}(\lambda) = 0, \qquad \gamma_r P_{\mathrm{cyl}}(\lambda) = \mathrm{Id}_{Y_r},$$
$$\Pi_{>,\tau^+}\gamma_0 P_{\mathrm{cyl}}(\lambda) = 0, \qquad \Pi_{<,\tau^-}\gamma_0(\partial_u + B)P_{\mathrm{cyl}}(\lambda) = 0.$$

We define the operator $Q_{\mathrm{cyl}}(\lambda) : C^\infty(Y_r) \to C^\infty(Y_r)$ by $Q_{\mathrm{cyl}}(\lambda) = \gamma_r \partial_u P_{\mathrm{cyl}}(\lambda)$ and define

$$\begin{aligned} R_{\mathrm{cyl}}(\lambda) &:= \Pi_{>,\sigma^+} Q_{\mathrm{cyl}}(\lambda) \Pi_{>,\sigma^+} + |B| + \sigma^- \\ &= \Pi_{>,\sigma^+}(Q_{\mathrm{cyl}}(\lambda) + |B|)\Pi_{>,\sigma^+} + |B|\Pi_< + \sigma^-. \end{aligned}$$

Direct computation shows that $\left(\Pi_{>,\sigma^+}(Q_{\mathrm{cyl}}(t) + |B|)\Pi_{>,\sigma^+}\right)$ is described as follows.

Lemma 4.1. *Suppose that $Bf = \mu f$ and $\widetilde{Q}_{\mathrm{cyl}}(\lambda) = \left(\Pi_{>,\sigma^+}(Q_{\mathrm{cyl}}(\lambda) + |B|)\Pi_{>,\sigma^+}\right)$.*
(1) *If $\mu > 0$,*

$$\widetilde{Q}_{\mathrm{cyl}}(\lambda) f = \left(\sqrt{\mu^2+\lambda} + \mu + \frac{2\sqrt{\mu^2+\lambda}\,\mathrm{e}^{-r\sqrt{\mu^2+\lambda}}}{\mathrm{e}^{r\sqrt{\mu^2+\lambda}} - \mathrm{e}^{-r\sqrt{\mu^2+\lambda}}}\right) f.$$

(2) *If $\mu = 0$ and $f \in \mathrm{Im}\,\sigma^+ \cap \mathrm{Im}\,\tau^-$,*

$$\widetilde{Q}_{\mathrm{cyl}}(\lambda) f = \left(\frac{\sqrt{\lambda}\left(\mathrm{e}^{r\sqrt{\lambda}} - \mathrm{e}^{-r\sqrt{\lambda}}\right)}{\mathrm{e}^{r\sqrt{\lambda}} + \mathrm{e}^{-r\sqrt{\lambda}}}\right) f.$$

(3) *If $\mu = 0$ and $f \in \mathrm{Im}\,\sigma^+ \cap (\mathrm{Im}\,\sigma^+ \cap \mathrm{Im}\,\tau^-)^\perp$,*

$$\widetilde{Q}_{\mathrm{cyl}}(\lambda) f = \left(\frac{\sqrt{\lambda}\left(\mathrm{e}^{r\sqrt{\lambda}} - \mathrm{e}^{-r\sqrt{\lambda}}\right)}{\mathrm{e}^{r\sqrt{\lambda}} + \mathrm{e}^{-r\sqrt{\lambda}}}\frac{I+\sigma}{2} + \frac{4\sqrt{\lambda}}{\mathrm{e}^{2r\sqrt{\lambda}} - \mathrm{e}^{-2r\sqrt{\lambda}}}\frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right) f.$$

Proof. (1) is straightforward. If $Bf = 0$ and $f \in \mathrm{Im}\,\sigma^+$, $P_{\mathrm{cyl}}(\lambda)(f)$ is given by

$$P_{\mathrm{cyl}}(\lambda)(f)(u,y) = \frac{\mathrm{e}^{\sqrt{\lambda}u} + \mathrm{e}^{-\sqrt{\lambda}u}}{\mathrm{e}^{r\sqrt{\lambda}} + \mathrm{e}^{-r\sqrt{\lambda}}}\frac{I+\sigma}{2} f(y) + \frac{2\left(\mathrm{e}^{\sqrt{\lambda}(u-r)} - \mathrm{e}^{-\sqrt{\lambda}(u-r)}\right)}{\mathrm{e}^{2r\sqrt{\lambda}} - \mathrm{e}^{-2r\sqrt{\lambda}}}\frac{I+\tau}{2}\frac{I+\sigma}{2} f(y).$$

Taking the derivative of $P_{\mathrm{cyl}}(\lambda)(f)(u,y)$ with respect to u at $u = r$ gives (2) and (3). □

Corollary 4.2.

$$\Pi_{>,\sigma^+}\left(Q_{\mathrm{cyl}}(\lambda) + |B|\right)\Pi_{>,\sigma^+} = \Pi_{>,\sigma^+}\left(\sqrt{B^2+\lambda} + |B|\right)\Pi_{>,\sigma^+} + \text{ a smoothing operator}.$$

Proceeding as in the proof of Theorem 1.1, we obtain the following result.

Lemma 4.3.

$$\begin{aligned} &\log \mathrm{Det}\left(-\partial_u^2 + B^2 + \lambda\right)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} - \log \mathrm{Det}\left(-\partial_u^2 + B^2 + \lambda\right)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r} \\ &\quad = \sum_{j=0}^{\nu-1} a_j \lambda^j + \log \mathrm{Det}\left(\Pi_{>,\sigma^+}(Q_{\mathrm{cyl}}(\lambda) + |B|)\Pi_{>,\sigma^+}\right) + \log \mathrm{Det}^*(|B|\Pi_<). \end{aligned}$$

Using the same argument as in the proof of Theorem 1.1, it is not difficult to see that the zero-coefficients in the asymptotic expansions, for $\lambda \to \infty$, of $\log \mathrm{Det}\left(-\partial_u^2 + B^2 + \lambda\right)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}}$, $\log \mathrm{Det}\left(-\partial_u^2 + B^2 + \lambda\right)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}$ and $\log \mathrm{Det}\left(\Pi_{>,\sigma^+}(Q_{\mathrm{cyl}}(\lambda) + |B|)\Pi_{>,\sigma^+}\right)$ are zeros, which implies that $a_0 + \log \mathrm{Det}^*(|B|\Pi_<) = 0$. We next discuss the behavior of each term in Lemma 4.3 for $\lambda \to 0$. We define $\mathfrak{M} = \mathrm{Im}\,\sigma^+ \cap (\mathrm{Im}\,\sigma^+ \cap \mathrm{Im}\,\tau^-)^\perp$,

$$k_+ = \dim(\mathrm{Im}\,\sigma^+ \cap \mathrm{Im}\,\tau^-) \quad \text{and} \quad \frac{l}{2} - k_+ = \dim \mathfrak{M}.$$

The equality (4.1) and the invertibility of $(-\partial_u^2 + B^2)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}$ imply that

$$\begin{aligned}
\log \mathrm{Det}\left(-\partial_u^2 + B^2 + \lambda\right)_{\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} &= k_+ \log\lambda + \log \mathrm{Det}^*\left(-\partial_u^2 + B^2\right)_{\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} + o(\lambda),\\
\log \mathrm{Det}\left(-\partial_u^2 + B^2 + \lambda\right)_{\Pi_{>,\tau^+},\gamma_r} &= \log \mathrm{Det}\left(-\partial_u^2 + B^2\right)_{\Pi_{>,\tau^+},\gamma_r} + o(\lambda).
\end{aligned} \tag{4.3}$$

Lemma 4.1 shows that

$$\begin{aligned}
\log \mathrm{Det}&\left(\Pi_{>,\sigma^+}(Q_{\mathrm{cyl}}(\lambda) + |B|)\Pi_{>,\sigma^+}\right) = \log \mathrm{Det}^*\left(\left(2|B| + \frac{2|B|\mathrm{e}^{-r|B|}}{\mathrm{e}^{r|B|} - \mathrm{e}^{-r|B|}}\right)\Pi_>\right)\\
&+ k_+(\log r + \log\lambda) + \log\det\left(\left(\frac{1}{r}\frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right)\Big|_{\mathfrak{M}}\right) + o(\lambda)\\
&= \frac{1}{2}\log \mathrm{Det}^*(2|B|) + \frac{1}{2}\log \det{}_{\mathrm{Fr}}\left(I + \frac{\mathrm{e}^{-r|B|}}{\mathrm{e}^{r|B|} - \mathrm{e}^{-r|B|}}\mathrm{pr}_{(\ker B)^\perp}\right)\\
&+ \left(2k_+ - \frac{l}{2}\right)\log r + k_+\log\lambda + \log\det\left(\frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right)\Big|_{\mathfrak{M}} + o(\lambda).
\end{aligned} \tag{4.4}$$

Lemma 4.4.

$$\det\left(\frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right)\Big|_{\mathfrak{M}} = \left|\det{}^*\left(\frac{\sigma+\tau}{2}\right)\right|,$$

where $\det^*\left(\frac{\sigma+\tau}{2}\right) = \det\left(\frac{\sigma+\tau}{2} + \mathrm{pr}_{\ker\frac{\sigma+\tau}{2}}\right)$.

Proof. If we define $\Sigma^\pm = (\mathrm{Im}\,\sigma^\pm \cap \mathrm{Im}\,\tau^\mp)$, we have

$$\det\left(\left(\frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right)\Big|_{\mathfrak{M}}\right) = \det\left(\frac{I-\sigma}{2} + \mathrm{pr}_{\Sigma^+} + \frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right).$$

Since $\det G = 1$ and $G \circ \mathrm{pr}_{\Sigma^+} = \mathrm{pr}_{\Sigma^-} \circ G$, we have

$$\det\left(\frac{I-\sigma}{2} + \mathrm{pr}_{\Sigma^+} + \frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right) = \det\left(\frac{I+\sigma}{2} + \mathrm{pr}_{\Sigma^-} + \frac{I-\sigma}{2}\frac{I-\tau}{2}\frac{I-\sigma}{2}\right),$$

and hence,

$$\begin{aligned}
\left(\det\left(\frac{I-\sigma}{2} + \mathrm{pr}_{\Sigma^+} + \frac{I+\sigma}{2}\frac{I+\tau}{2}\frac{I+\sigma}{2}\right)\right)^2 &= \det\left(\mathrm{pr}_{\Sigma^+} + \mathrm{pr}_{\Sigma^-} + \left(\frac{\sigma+\tau}{2}\right)^2\right)\\
&= \det\left(\mathrm{pr}_{\ker(\sigma+\tau)} + \left(\frac{\sigma+\tau}{2}\right)^2\right).
\end{aligned}$$

Since the determinant of an operator that we want to compute is positive, the result follows. $\square$

Since $\log \mathrm{Det}^*(2|B|) = \log 2 \cdot \zeta_{B^2}(0) + \frac{1}{2}\log \mathrm{Det}^* B^2$, Lemma 4.3 and (4.3) and (4.4) lead to the following result.

Theorem 4.5.

$$\log \mathrm{Det}^*\left(-\partial_u^2+B^2\right)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} - \log \mathrm{Det}\left(-\partial_u^2+B^2\right)_{N_{0,r},\Pi_{>,\tau^+},\gamma_r}$$
$$= \left(2k_+ - \frac{l}{2}\right)\log r + \frac{1}{2}\log 2\cdot\zeta_{B^2}(0) + \frac{1}{4}\log \mathrm{Det}^* B^2 + \log\left|\det{}^*\left(\frac{\sigma+\tau}{2}\right)\right|$$
$$+\frac{1}{2}\log \det{}_{\mathrm{Fr}}\left(I+\frac{\mathrm{e}^{-r|B|}}{\mathrm{e}^{r|B|}-\mathrm{e}^{-r|B|}}\mathrm{pr}_{(\ker B)^\perp}\right).$$

Corollary 4.6.

$$\log \mathrm{Det}^*\left(-\partial_u^2+B^2\right)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} - \log \mathrm{Det}\left(-\partial_u^2+B^2\right)_{N_{0,r},\gamma_0,\gamma_r}$$
$$= (2k_+ - l)\log r + \log 2\cdot\zeta_{B^2}(0) + \frac{1}{2}\log \mathrm{Det}^* B^2 + \log\left|\det{}^*\left(\frac{\sigma+\tau}{2}\right)\right|$$
$$+\log \det{}_{\mathrm{Fr}}\left(I+\frac{\mathrm{e}^{-r|B|}}{\mathrm{e}^{r|B|}-\mathrm{e}^{-r|B|}}\mathrm{pr}_{(\ker B)^\perp}\right).$$

It is a well-known fact (cf. [16] or [23]) that

$$\log \mathrm{Det}(-\partial_u^2+B^2)_{N_{0,r},\gamma_0,\gamma_r} = l\cdot\log 2 + l\cdot\log r + \alpha_1\cdot r - \frac{1}{2}\log \mathrm{Det}^* B^2 + \log \det{}_{\mathrm{Fr}}(I-\mathrm{e}^{-2r|B|}\mathrm{pr}_{(\ker B)^\perp}), \tag{4.5}$$

where α_1 is the constant defined in (1.14). For any positive real number μ, we note that

$$\left(1-\mathrm{e}^{-2r\mu}\right)\left(1+\frac{\mathrm{e}^{-r\mu}}{\mathrm{e}^{r\mu}-\mathrm{e}^{-r\mu}}\right)=1.$$

Corollary 4.6 and (4.5) with this observation lead to

$$\log \mathrm{Det}^*\left(-\partial_u^2+B^2\right)_{N_{0,r},\Pi_{>,\tau^+},\Pi_{<,\sigma^-}} = \alpha_1\cdot r + 2k_+\log r + \log 2\cdot(\zeta_{B^2}(0)+l) + \log\left|\det{}^*\left(\frac{\sigma+\tau}{2}\right)\right|,$$

which completes the proof of the first equality in Theorem 1.5.

To prove the second equality, we play the same game with $(-\partial_u^2+B^2+\lambda)_{N_{0,r},\gamma_0,(\partial_u+|B|)}$ and $(-\partial_u^2+B^2+\lambda)_{N_{0,r},\gamma_0,\gamma_r}$. We define $R_{(\partial_u+|B|)}(\lambda): C^\infty(Y_r)\to C^\infty(Y_r)$ corresponding to $R_P(\lambda)$ in Theorem 1.1 as follows:

$$R_{(\partial_u+|B|)}(\lambda) = \gamma_r(\partial_u+|B|)P_{\gamma_r}(\lambda) = Q_1(\lambda)+|B|,$$

where $P_{\gamma_r}(\lambda)$ is the Poisson operator defined on Y_r characterized as follows:

$$(-\partial_u^2+B^2+\lambda)P_{\gamma_r}(\lambda)=0,\qquad \gamma_0 P_{\gamma_r}(\lambda)=0,\qquad \gamma_r P_{\gamma_r}(\lambda)=\mathrm{Id}.$$

Then proceeding as in the proof of Theorem 1.1, we have the following equality:

$$\log \mathrm{Det}(-\partial_u^2+B^2+\lambda)_{N_{0,r},\gamma_0,(\partial_u+|B|)} - \log \mathrm{Det}(-\partial_u^2+B^2+\lambda)_{N_{0,r},\gamma_0,\gamma_r} = \sum_{j=0}^{\nu-1} a_j\lambda^j + \log \mathrm{Det}(Q_1(\lambda)+|B|).$$

For $\lambda\to\infty$, the zero-coefficients in the asymptotic expansions of $\log \mathrm{Det}(-\partial_u^2+B^2+\lambda)_{N_{0,r},\gamma_0,(\partial_u+|B|)}$, $\log \mathrm{Det}(-\partial_u^2+B^2+\lambda)_{N_{0,r},\gamma_0,\gamma_r}$ and $\log \mathrm{Det}(Q_1(\lambda)+|B|)$ are zeros, which implies that $a_0=0$. Moreover, $(-\partial_u^2+B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)}$, $(-\partial_u^2+B^2)_{N_{0,r},\gamma_0,\gamma_r}$ and $(Q_1+|B|)$ are invertible operators, and hence

$$\log \mathrm{Det}(-\partial_u^2+B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)} - \log \mathrm{Det}(-\partial_u^2+B^2)_{N_{0,r},\gamma_0,\gamma_r} = \log \mathrm{Det}(Q_1+|B|).$$

Since $Q_1=\frac{1}{r}\mathrm{pr}_{\ker B}+|B|+\frac{2|B|\mathrm{e}^{-r|B|}}{\mathrm{e}^{r|B|}-\mathrm{e}^{-r|B|}}\mathrm{pr}_{(\ker B)^\perp}$ (cf. (1.12)), we have the following result. This result and (4.5) yield the second equality of Theorem 1.5.

Theorem 4.7.

$$\log \operatorname{Det}(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,(\partial_u+|B|)} - \log \operatorname{Det}(-\partial_u^2 + B^2)_{N_{0,r},\gamma_0,\gamma_r}$$
$$= -l \cdot \log r + \log 2 \cdot \zeta_{B^2}(0) + \frac{1}{2} \log \operatorname{Det}^* B^2 + \log \det_{\mathrm{Fr}} \left(I + \frac{e^{-r|B|}}{e^{r|B|} - e^{-r|B|}} \mathrm{pr}_{(\ker B)^\perp} \right).$$

5. The proof of Theorem 1.6

In this section, we are going to prove Theorem 1.6. For simplicity, we denote $(-\partial_u^2 + B^2)_{N_{0,\infty},\gamma_0}$, $(-\partial_u^2 + B^2)_{N_{0,\infty},\Pi_{>,\tau^+}}$ by Δ_{∞,γ_0}, $\Delta_{\infty,\Pi_{>,\tau^+}}$, respectively. Then Eq. (1.15) implies that

$$\log \operatorname{Det}\left(D^2_{M_1,\infty}, \Delta_{\infty,\Pi_{>,\tau^+}}\right) - \log \operatorname{Det}^*\left(D^2_{M_1,\Pi_{<,\sigma^-}}\right) = -\log 2 \cdot (\zeta_{B^2}(0) + l) - \log \det A_1$$
$$+ \log \operatorname{Det}\left(\Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}\right) + \log \operatorname{Det}^*(Q_1 + |B|) + \log \operatorname{Det} D^2_{M_1,\gamma_0} - \log \operatorname{Det}^*\left(D^2_{M_1,\Pi_{<,\sigma^-}}\right). \quad (5.1)$$

We now compute the relative zeta-determinant $\log \operatorname{Det}\left(\Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}\right)$. The relative zeta function $\zeta(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}})$ is defined by

$$\zeta(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \int_{N_{0,\infty}} \Big(e^{-t\Delta_{\infty,\gamma_0}}(t,(u,y),(u,y))$$
$$- e^{-t\Delta_{\infty,\Pi_{>,\tau^+}}}(t,(u,y),(u,y)) \Big)\, dvol(y) \mathrm{d}u \mathrm{d}t$$

and $\log \operatorname{Det}\left(\Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}\right) = -\frac{d}{ds}|_{s=0} \zeta(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}})$. It is a well-known fact (cf. [1] or [3]) that the heat kernels $e^{-t\Delta_{\infty,\gamma_0}}(t,(u,y),(v,z))$ and $e^{-t\Delta_{\infty,\Pi_{>,\tau^+}}}(t,(u,y),(v,z))$ are given as follows:

$$e^{-t\Delta_{\infty,\gamma_0}}(t,(u,y),(v,z)) = \sum_{\mu_j \in \operatorname{Spec}(B)} \frac{e^{-\mu_j^2 t}}{\sqrt{4\pi t}} \left\{ e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}} \right\} \varphi_j(y) \otimes \varphi_j(z).$$

$$e^{-t\Delta_{\infty,\Pi_{>,\tau^+}}}(t,(u,y),(v,z)) = \sum_{0<\mu_j \in \operatorname{Spec}(B)} \frac{e^{-\mu_j^2 t}}{\sqrt{4\pi t}} \left\{ e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}} \right\} \varphi_j(y) \otimes \varphi_j(z)$$
$$+ \sum_{\phi_j \in \tau^-} \frac{1}{\sqrt{4\pi t}} \left\{ e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}} \right\} \phi_j(y) \otimes \phi_j(z)$$
$$+ \sum_{\psi_j \in \tau^+} \frac{1}{\sqrt{4\pi t}} \left\{ e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}} \right\} \psi_j(y) \otimes \psi_j(z)$$
$$+ \sum_{0<\mu_j \in \operatorname{Spec}(B)} \left\{ \frac{e^{-\mu_j^2 t}}{\sqrt{4\pi t}} \left\{ e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}} \right\} \right.$$
$$\left. - \mu_j e^{\mu_j(u+v)} \operatorname{erfc}\left(\frac{u+v}{2\sqrt{t}} + \mu_j \sqrt{t} \right) \right\} G\varphi_j(y) \otimes G\varphi_j(z),$$

where $B\varphi_j = \mu_j \varphi_j$ and $\operatorname{erfc}(x)$ is the error function defined by $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \mathrm{d}t$. Then direct computation shows that

$$\operatorname{Tr}\left(e^{-t\Delta_{\infty,\gamma_0}} - e^{-t\Delta_{\infty,\Pi_{>,\tau^+}}} \right) = -\frac{l}{4} - \frac{1}{2} \sum_{\mu_j > 0} \operatorname{erfc}(\mu_j \sqrt{t}).$$

According to [22] we split $\zeta(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}})$ into two parts:

$$\zeta(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}) = \zeta_1(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}) + \zeta_2(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}),$$

where

$$\zeta_1(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \mathrm{Tr}\left(\mathrm{e}^{-t\Delta_{\infty,\gamma_0}} - \mathrm{e}^{-t\Delta_{\infty,\Pi_{>,\tau^+}}}\right) \mathrm{d}t,$$

$$\zeta_2(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}) = \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \mathrm{Tr}\left(\mathrm{e}^{-t\Delta_{\infty,\gamma_0}} - \mathrm{e}^{-t\Delta_{\infty,\Pi_{>,\tau^+}}}\right) \mathrm{d}t.$$

For Re $s > 0$,

$$\zeta_1(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}) = -\frac{l}{4}\frac{1}{\Gamma(s+1)} - \frac{1}{2}\frac{1}{\Gamma(s+1)} \sum_{\mu_j>0} \mathrm{erfc}(\mu_j) - \frac{1}{4\sqrt{\pi}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \zeta_{B^2}(s)$$
$$+ \frac{1}{\sqrt{4\pi}} \frac{1}{\Gamma(s+1)} \sum_{\mu_j>0} \int_1^\infty t^{s-\frac{1}{2}} \mu_j \mathrm{e}^{-t\mu_j^2} \mathrm{d}t. \tag{5.2}$$

For Re $s < 0$,

$$\zeta_2(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}) = \frac{l}{4}\frac{1}{\Gamma(s+1)} + \frac{1}{2}\frac{1}{\Gamma(s+1)} \sum_{\mu_j>0} \mathrm{erfc}(\mu_j)$$
$$- \frac{1}{\sqrt{4\pi}} \frac{1}{\Gamma(s+1)} \sum_{\mu_j>0} \int_1^\infty t^{s-\frac{1}{2}} \mu_j \mathrm{e}^{-t\mu_j^2} \mathrm{d}t. \tag{5.3}$$

Since the last terms in (5.2) and (5.3) are entire functions, the right hand sides of (5.2) and (5.3) give the meromorphic continuations of $\zeta_1(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}})$ and $\zeta_2(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}})$ to the whole complex plane, having regular values at $s = 0$. Therefore, we have

$$\zeta(s, \Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}) = -\frac{1}{4\sqrt{\pi}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \zeta_{B^2}(s).$$

Since $\Gamma'(\frac{1}{2}) = -\sqrt{\pi}(\gamma + 2\log 2)$ (cf. p. 15 in [20]) for $\gamma = -\Gamma'(1)$ the Euler constant, we have the following result.

Lemma 5.1.

$$\log \mathrm{Det}\left(\Delta_{\infty,\gamma_0}, \Delta_{\infty,\Pi_{>,\tau^+}}\right) = -\frac{1}{2}\log 2 \cdot \zeta_{B^2}(0) - \frac{1}{4} \log \mathrm{Det}^* B^2.$$

The above lemma together with (5.1) leads to the following equality:

$$\log \mathrm{Det}\left(D^2_{M_{1,\infty}}, (-\partial_u^2 + B^2)_{N_{0,\infty},\Pi_{>,\tau^+}}\right) - \log \mathrm{Det}(D^2_{M_1,\Pi_{<,\sigma^-}}) = -\log\det A_1 - \frac{1}{4}\log \mathrm{Det}^* B^2$$
$$- \log 2 \cdot \left(\frac{3}{2}\zeta_{B^2}(0) + l\right) + \log \mathrm{Det}^*(Q_1 + |B|) + \log \mathrm{Det}\, D^2_{M_1,\gamma_0} - \log \mathrm{Det}^*\left(D^2_{M_1,\Pi_{<,\sigma^-}}\right). \tag{5.4}$$

Next, we are going to analyze the term $\log \mathrm{Det}^*(Q_1 + |B|)$. Let $L_{2,M_{1,\infty}}$, $L^{\mathrm{ext}}_{2,M_{1,\infty}}$ be the spaces of all L^2- and extended L^2-solutions of $D_{M_{1,\infty}}$ on $M_{1,\infty}$. Then it is not difficult to show (cf. [13] or [14]) that

$$\ker(Q_1 + |B|) = \left\{\phi|_Y \mid \phi \in \left(L_{2,M_{1,\infty}} + L^{\mathrm{ext}}_{2,M_{1,\infty}}\right)\right\} = \mathrm{Im}\,\mathfrak{C}_1 \cap \mathrm{Im}\,\Pi_{>,C(0)^+}. \tag{5.5}$$

We define $\dim\ker(Q_1+|B|)=q$. Using (5.5) we decompose $L^2(Y,E|_Y)$ as

$$L^2(Y,E|_Y)=\ker(Q_1+|B|)\oplus\left(\mathrm{Im}\,(I-\mathfrak{C}_1)+\mathrm{Im}\left(I-\Pi_{>,C(0)^+}\right)\right). \tag{5.6}$$

Let K_1, $T_0 : L^2(E_Y^+)\to L^2(E_Y^-)$ be unitary maps whose graphs are $\mathrm{Im}\,\mathfrak{C}_1$, $\mathrm{Im}\,\Pi_{>,C(0)^+}$, respectively. We now consider $\left(Q_1+|B|+\mathrm{pr}_{\mathrm{Im}\,C(0)^-}\right)$ rather than $(Q_1+|B|)$. Using (3.4) with $K=K_1$ and $T=T_0$, we have

$$\begin{aligned}\left(Q_1+|B|+\mathrm{pr}_{\mathrm{Im}\,C(0)^-}\right)(I-K_1)\,x &= (I-\mathfrak{C}_1)\,(Q_1+B)\,(I-\mathfrak{C}_1)\,(I-K_1)x\\ &\quad+\left(I-\Pi_{>,C(0)^+}\right)\left(|B|-B+\mathrm{pr}_{\mathrm{Im}\,C(0)^-}\right)\\ &\quad\times\left(I-\Pi_{>,C(0)^+}\right)(I-T_0)\frac{I+T_0^{-1}K_1}{2}x,\\ \left(Q_1+|B|+\mathrm{pr}_{\mathrm{Im}\,C(0)^-}\right)(I-T_0)\,y &= (I-\mathfrak{C}_1)\,(Q_1+B)\,(I-\mathfrak{C}_1)\,(I-K_1)\frac{I+K_1^{-1}T_0}{2}y\\ &\quad+\left(I-\Pi_{>,C(0)^+}\right)\left(|B|-B+\mathrm{pr}_{\mathrm{Im}\,C(0)^-}\right)\left(I-\Pi_{>,C(0)^+}\right)(I-T_0)y.\end{aligned}$$

Recall that $\ker(Q_1+|B|)=\ker\left(Q_1+|B|+\mathrm{pr}_{\mathrm{Im}\,C(0)^-}\right)=\{(I+K_1)x\mid K_1x=T_0x\}$ and we denote it by H. We now define a subspace $\widetilde{H}_-$ of $\mathrm{Im}\,(I-\mathfrak{C}_1)\oplus\mathrm{Im}\left(I-\Pi_{>,C(0)^+}\right)$ by

$$\widetilde{H}_-=\{(I-K_1)x,-(I-T_0)x\mid K_1x=T_0x\},$$

and consider the following diagram:

$$\begin{array}{ccc}\mathrm{Im}\,(I-\mathfrak{C}_1)\oplus\mathrm{Im}\left(I-\Pi_{>,C(0)^+}\right) & \xrightarrow{\widetilde{R}} & \mathrm{Im}\,(I-\mathfrak{C}_1)\oplus\mathrm{Im}\left(I-\Pi_{>,C(0)^+}\right)\\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi}\\ \left(\mathrm{Im}\,(I-\mathfrak{C}_1)+\mathrm{Im}\left(I-\Pi_{>,C(0)^+}\right)\right)\oplus\widetilde{H}_- & \xrightarrow{\widetilde{Q}} & \left(\mathrm{Im}\,(I-\mathfrak{C}_1)+\mathrm{Im}\left(I-\Pi_{>,C(0)^+}\right)\right)\oplus\widetilde{H}_-,\end{array} \tag{5.7}$$

where Φ, $\widetilde{Q}$ and $\widetilde{R}$ are defined as follows:

$$\begin{aligned}&\Phi\,((I-K_1)x,(I-T_0)y)=\Big((I-K_1)x+(I-T_0)y,\mathrm{pr}_{\widetilde{H}_-}\,((I-K_1)x,(I-T_0)y)\Big),\\ &\widetilde{Q}\,(a,b)=\Big(\left(Q_1+|B|+\mathrm{pr}_{\mathrm{Im}\,C(0)^-}\right)(a)\,,\mathrm{pr}_{\widetilde{H}_-}\widetilde{R}\,\Phi^{-1}(a,b)\Big),\\ &\widetilde{R}=\begin{pmatrix}\mathfrak{S}_1 & \mathfrak{S}_1(I-K_1)\frac{I+K_1^{-1}T_0}{2}(I-T_0)^{-1}\\ \mathfrak{S}_2(I-T_0)\frac{I+T_0^{-1}K_1}{2}(I-K_1)^{-1} & \mathfrak{S}_2\end{pmatrix}\mathrm{pr}_{(\widetilde{H}_-)^{\perp}}+r_0\mathrm{pr}_{\widetilde{H}_-}\\ &\quad=\begin{pmatrix}\mathfrak{S}_1 & 0\\ 0 & \mathfrak{S}_2\end{pmatrix}\begin{pmatrix}I & (I-K_1)\frac{I+K_1^{-1}T_0}{2}(I-T_0)^{-1}\\ (I-T_0)\frac{I+T_0^{-1}K_1}{2}(I-K_1)^{-1} & I\end{pmatrix}+r_0\mathrm{pr}_{\widetilde{H}_-},\end{aligned}$$

where $\mathfrak{S}_1=(I-\mathfrak{C}_1)\,(Q_1+B)\,(I-\mathfrak{C}_1)$, $\mathfrak{S}_2=\Pi_{<,C(0)^-}\left(|B|-B+\mathrm{pr}_{\mathrm{Im}\,C(0)^-}\right)\Pi_{<,C(0)^-}$, and r_0 is a positive real number such that $r_0\notin\mathrm{Spec}(Q_1+|B|+\mathrm{pr}_{\mathrm{Im}\,C(0)^-})$. Then all maps are invertible and the diagram (5.7) commutes. In the same way as in Section 3, we have

$$\begin{aligned}
\log \operatorname{Det} \widetilde{Q} &= q \log r_0 + \log \operatorname{Det}^* \left(Q_1 + |B| + \mathrm{pr}_{\mathrm{Im}\ C(0)^-}\right) \\
&= \log \operatorname{Det}\left((I - \mathfrak{C}_1)(Q_1 + B)(I - \mathfrak{C}_1)\right) + \log \operatorname{Det}^* (2|B|\Pi_<) \\
&\quad + \log \left| \det{}^* \left(\frac{I - K_1^{-1} T_0}{2} \right) \right|^2 + q \log 2 + q \log r_0 - q \log 2 \\
&\quad + \log \det \left(\mathrm{pr}_{GH} \left((Q_1 + B)^{-1} + (|B| - B + \mathrm{pr}_{\mathrm{Im}\ C(0)^-})^{-1} \right) \mathrm{pr}_{GH} \right) \\
&= q \log r_0 + \log \operatorname{Det} D^2_{M_1, \mathfrak{C}_1} - \log \operatorname{Det} D^2_{M_1, \gamma_0} + \frac{1}{2} \log 2 \cdot \zeta_{B^2}(0) + \frac{1}{4} \log \operatorname{Det}^* B^2 \\
&\quad + \log \left| \det{}^* \left(\frac{I - K_1^{-1} T_0}{2} \right) \right|^2 + \log \det \left(\mathrm{pr}_{GH} \left((Q_1 + B)^{-1} + (|B| - B + \mathrm{pr}_{\mathrm{Im}\ C(0)^-})^{-1} \right) \mathrm{pr}_{GH} \right).
\end{aligned} \tag{5.8}$$

We next discuss the relation between $\log \operatorname{Det}^* \left(Q_1 + |B| + \mathrm{pr}_{\mathrm{Im}\ C(0)^-}\right)$ and $\log \operatorname{Det}^* (Q_1 + |B|)$. Since $\ker(Q_1 + |B|) = \ker(Q_1 + |B| + \mathrm{pr}_{\mathrm{Im}\ C(0)^-})$, we have

$$\begin{aligned}
\log \operatorname{Det}^* \left(Q_1 + |B| + \mathrm{pr}_{\mathrm{Im}\ C(0)^-}\right) &= \log \operatorname{Det} \left(Q_1 + |B| + \mathrm{pr}_{\mathrm{Im}\ C(0)^-} + \mathrm{pr}_{\ker(Q_1 + |B|)}\right) \\
&= \log \operatorname{Det} \left(Q_1 + |B| + \mathrm{pr}_{\ker(Q_1 + |B|)}\right) \\
&\quad + \log \det{}_{\mathrm{Fr}} \left(I + \left(Q_1 + |B| + \mathrm{pr}_{\ker(Q_1 + |B|)}\right)^{-1} \mathrm{pr}_{\mathrm{Im}\ C(0)^-} \right) \\
&= \log \operatorname{Det} \left(Q_1 + |B| + \mathrm{pr}_{\ker(Q_1 + |B|)}\right) \\
&\quad + \log \det{}_{\mathrm{Fr}} \left(I + \mathrm{pr}_{\mathrm{Im}\ C(0)^-} \left(Q_1 + |B| + \mathrm{pr}_{\ker(Q_1 + |B|)}\right)^{-1} \mathrm{pr}_{\mathrm{Im}\ C(0)^-} \right).
\end{aligned} \tag{5.9}$$

Let $\{f_1, f_2, \ldots, f_{\frac{l}{2}}\}$ be an orthonormal basis for $\mathrm{Im}\ C(0)^+$. Then $\{Gf_1, Gf_2, \ldots, Gf_{\frac{l}{2}}\}$ is an orthonormal basis for $\mathrm{Im}\ C(0)^-$. Defining $\left(Q_1 + |B| + \mathrm{pr}_{\ker(Q_1 + |B|)}\right)^{-1} Gf_i = F_i$, we have

$$Gf_i = \left(Q_1 + |B| + \mathrm{pr}_{\ker(Q_1 + |B|)}\right) F_i = (Q_1 + |B|)\, F_i,$$

and hence

$$\langle \left(Q_1 + |B| + \mathrm{pr}_{\ker(Q_1 + |B|)}\right)^{-1} Gf_i, Gf_j \rangle_Y = \langle F_i, (Q_1 + |B|)\, F_j \rangle_Y = \langle (Q_1 + |B|)\, F_i, F_j \rangle_Y. \tag{5.10}$$

Choose $\phi_i \in C^\infty(M_1)$ such that $D^2_{M_1} \phi_i = 0$ and $\phi_i|_Y = F_i$. Then

$$\begin{aligned}
0 = \langle D^2_{M_1} \phi_i, \phi_j \rangle_{M_1} &= \langle D_{M_1} \phi_i, D_{M_1} \phi_j \rangle_{M_1} - \langle (D_{M_1} \phi_i)|_Y, (G\phi_j)|_Y \rangle_Y \\
&= \langle D_{M_1} \phi_i, D_{M_1} \phi_j \rangle_{M_1} - \langle (Q_1 + B)\, F_i, F_j \rangle_Y,
\end{aligned}$$

which shows that $\langle (Q_1 + B)\, F_i, F_j \rangle_Y = \langle D_{M_1} \phi_i, D_{M_1} \phi_j \rangle_{M_1}$ and hence

$$\begin{aligned}
\langle (Q_1 + |B|)\, F_i, F_j \rangle_Y &= \langle (Q_1 + B)\, F_i, F_j \rangle_Y + \langle (|B| - B)\, F_i, F_j \rangle_Y \\
&= \langle D_{M_1} \phi_i, D_{M_1} \phi_j \rangle_{M_1} + \langle (|B| + B)\, GF_i, GF_j \rangle_Y \\
&= \langle D_{M_1} \phi_i, D_{M_1} \phi_j \rangle_{M_1} + \langle (|B| + B)\, GF_i, (|B| + B)^{-1} (|B| + B)\, GF_j \rangle_Y.
\end{aligned} \tag{5.11}$$

We note that

$$\begin{aligned}
\left(D_{M_1} \phi_i\right)|_Y &= (G(\partial_u + B)\phi_i)|_Y = G\,(Q_1 + B)\, F_i \\
&= G\,(Q_1 + |B|)\, F_i - G\,(|B| - B)\, F_i = G(Gf_i) - (|B| + B)\, GF_i \\
&= -\left(f_i + (|B| + B)\, GF_i\right),
\end{aligned} \tag{5.12}$$

which shows that $D_{M_1}(-\phi_i)$ can be extended to an extended L^2-solution of $D_{M_{1,\infty}}$ on $M_{1,\infty}$ whose limiting value is f_i.

Let $\{\psi_1, \ldots, \psi_{q'}\}$ be an orthonormal basis for L^2-solutions of $D_{M_{1,\infty}}$ and $\psi_{q'+j}$ $(1 \le j \le \frac{l}{2})$ be the extended L^2-solution of $D_{M_{1,\infty}}$ on $M_{1,\infty}$ which is the extension of $D_{M_1}(-\phi_j)$. Then

$$\{\psi_1, \psi_2, \ldots, \psi_{q'}, \psi_{q'+1}, \ldots, \psi_{q'+\frac{l}{2}}\}, \quad q = q' + \frac{l}{2} \tag{5.13}$$

is a basis for $\left(L_{2,M_{1,\infty}} + L^{\text{ext}}_{2,M_{1,\infty}}\right)$. We define $\psi_{i,L^2} = \psi_i$ for $1 \le i \le q'$ and for $1 \le j \le \frac{l}{2}$

$$\psi_{q'+j,L^2} = \begin{cases} D_{M_1}(-\phi_j) & \text{on } M_1 \\ (|B| + B)\, GF_j & \text{on } Y \times [0, \infty). \end{cases} \tag{5.14}$$

Then, (5.11) and (5.12) show that

$$\begin{aligned} \langle (Q_1 + |B|)\, F_i, F_j \rangle_Y &= \langle D_{M_1}\phi_i, D_{M_1}\phi_j \rangle_{M_1} + \langle (|B| + B)\, GF_i, (|B| + B)\, GF_j \rangle_{Y\times[0,\infty)} \\ &= \langle \psi_{q'+i,L^2}, \psi_{q'+j,L^2} \rangle_{M_{1,\infty}}. \end{aligned} \tag{5.15}$$

Moreover, for $1 \le i \le q'$, $1 \le j \le \frac{l}{2}$,

$$\begin{aligned} \langle \psi_i, \psi_{q'+j,L^2} \rangle_{M_{1,\infty}} &= \langle \psi_i, D_{M_1}(-\phi_j) \rangle_{M_1} + \langle \psi_i, (|B| + B)\, GF_j \rangle_{Y\times[0,\infty)} \\ &= \langle D_{M_1}\psi_i, -\phi_j \rangle_{M_1} - \langle \psi_i|_Y, G\phi_j|_Y \rangle_Y + \langle \psi_i|_Y, (|B| + B)^{-1}\, (|B| + B)\, GF_j \rangle_Y \\ &= -\langle \psi_i|_Y, GF_j \rangle_Y + \langle \psi_i|_Y, GF_j \rangle_Y = 0. \end{aligned} \tag{5.16}$$

Denoting by $\psi_{i,0}$ the limiting value of ψ_i, i.e. $\psi_{i,0} = 0$ for $0 \le i \le q'$ and $\psi_{q'+j,0} = f_j$ for $0 \le j \le \frac{l}{2}$, (5.10), (5.15) and (5.16) show that

$$\begin{aligned} &\log\det\left(I + \mathrm{pr}_{\mathrm{Im}\ C(0)^-}\left(Q_1 + |B| + \mathrm{pr}_{\ker(Q_1+|B|)}\right)^{-1} \mathrm{pr}_{\mathrm{Im}\ C(0)^-}\right) = \log\det(\langle \psi_{q'+i,0}, \psi_{q'+j,0} \rangle_Y \\ &\quad + \langle \psi_{q'+i,L^2}, \psi_{q'+j,L^2} \rangle_{M_{1,\infty}})_{1\le i,j\le\frac{l}{2}} \\ &= \log\det\left(\langle \psi_{i,0}, \psi_{j,0} \rangle_Y + \langle \psi_{i,L^2}, \psi_{j,L^2} \rangle_{M_{1,\infty}}\right)_{1\le i,j\le q}. \end{aligned} \tag{5.17}$$

Finally, we are going to analyze the last term in the last equality of (5.8). Let $\{h_1, \ldots, h_q\}$ be an orthonormal basis for $\mathrm{Im}\,\mathfrak{C}_1 \cap \mathrm{Im}\,\Pi_{>,C(0)^+}$. Then $\{Gh_1, \ldots, Gh_q\}$ is an orthonormal basis for $\mathrm{Im}\,(I - \mathfrak{C}_1) \cap \mathrm{Im}\,\Pi_{<,C(0)^-}$. Let $\varphi_1, \ldots, \varphi_q$ be elements in $\left(L_{2,M_{1,\infty}} + L^{\text{ext}}_{2,M_{1,\infty}}\right)$ such that $\varphi_i|_Y = h_i$. Then in the same way as in Lemma 3.1, we can show that

$$\langle (Q_1 + B)^{-1} Gh_i, Gh_j \rangle_Y = \langle \varphi_i|_{M_1}, \varphi_j|_{M_1} \rangle_{M_1}. \tag{5.18}$$

We denote by $\varphi_{i,0}$ the limiting value of φ_i, i.e. $\varphi_{i,0} = 0$ if φ_i is an L^2-solution. We define φ_{i,L^2} by (cf. (5.14))

$$\varphi_{i,L^2} = \begin{cases} \varphi_i & \text{on } M_1 \\ \varphi_i - \varphi_{i,0} & \text{on } Y \times [0, \infty). \end{cases}$$

Direct computation shows that

$$\langle (|B| - B + \mathrm{pr}_{\mathrm{Im}\ C(0)^-})^{-1} Gh_i, Gh_j \rangle_Y = \langle \varphi_{i,0}, \varphi_{j,0} \rangle_Y + \langle \varphi_{i,L^2}|_{N_{0,\infty}}, \varphi_{j,L^2}|_{N_{0,\infty}} \rangle_{N_{0,\infty}}, \tag{5.19}$$

where $N_{0,\infty} := [0, \infty) \times Y$. Hence, (5.18) and (5.19) imply that

$$\begin{aligned} \langle \left((Q_1 + B)^{-1} + (|B| - B + \mathrm{pr}_{\mathrm{Im}\ C(0)^-})^{-1}\right) Gh_i, Gh_j \rangle_Y &= \langle \varphi_{i,0}, \varphi_{j,0} \rangle_Y + \langle \varphi_{i,L^2}, \varphi_{j,L^2} \rangle_{M_{1,\infty}} \\ &=: \mathfrak{w}_{ij}, \end{aligned} \tag{5.20}$$

where we define $\mathfrak{W} = (\mathfrak{w}_{ij})$. As a basis for $\left(L_{2,M_{1,\infty}} + L^{\text{ext}}_{2,M_{1,\infty}}\right)$, we choose $\{\psi_1, \psi_2, \ldots, \psi_q\}$ defined in (5.13). Then $\psi_i = \sum_{j=1}^{q} c_{ij}\varphi_j$ for some $c_{ij} \in \mathbb{C}$ and we define a matrix $C = (c_{ij})$. Note that

$$\psi_i|_Y = \sum_{j=1}^{q} c_{ij}\varphi_j \Big|_Y = \sum_{j=1}^{q} c_{ij}h_j.$$

Setting $A_1 = \left(\langle \psi_i|_Y, \psi_j|_Y\rangle_Y\right)_{1\le i,j\le q}$, we have

$$A_1 = CC^*.$$

Then we have

$$\widetilde{V} := \left(\langle \psi_{i,0}, \psi_{j,0}\rangle_Y + \langle \psi_{i,L^2}, \psi_{j,L^2}\rangle_{M_{1,\infty}}\right)_{1\le i,j\le q} = C\mathfrak{W}C^*,$$

which shows that

$$\log\det\left(\mathrm{pr}_{GH}\left((Q_1 + B)^{-1} + (|B| - B + \mathrm{pr}_{\mathrm{Im}\ C(0)^-})^{-1}\right)\mathrm{pr}_{GH}\right) = -\log\det A_1 + \log\det \widetilde{V}. \tag{5.21}$$

Theorem 1.6 follows from (5.4), (5.8), (5.17) and (5.21) and Theorem 1.2.

Acknowledgements

The author was supported by KRF-2004-015-C00040 and thanks the Max Planck Institute for Mathematics for hospitality.

References

[1] M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry I, Math. Proc. Cambridge Philos. Soc. 77 (1975) 43–69.
[2] C. Bär, Zero sets of solutions to semilinear elliptic systems of first order, Invent. Math. 138 (1999) 183–202.
[3] B. Booß-Bavnbek, K. Wojciechowski, Elliptic Boundary Value Problems for Dirac Operators, Birkhäuser, Boston, 1993.
[4] J. Brüning, M. Lesch, On the η-invariant of certain nonlocal boundary value problems, Duke Math. J. 96 (1999) 425–468.
[5] D. Burghelea, L. Friedlander, T. Kappeler, Mayer–Vietoris type formula for determinants of elliptic differential operators, J. Funct. Anal. 107 (1992) 34–66.
[6] G. Carron, Determinant relatif et fonction Xi, Amer. J. Math. 124 (2002) 307–352.
[7] S. Cappell, R. Lee, E. Miller, Self-adjoint elliptic operators and manifold decompositions, Part I: Low eigenmodes and stretching, Comm. Pure Appl. Math. 96 (1996) 825–866.
[8] R. Forman, Functional determinants and geometry, Invent. Math. 88 (1987) 447–493.
[9] G. Grubb, Trace expansions for pseudodifferential boundary value problems for Dirac-type operators and more general systems, Ark. Mat. 37 (1999) 45–86.
[10] G. Grubb, R. Seeley, Weakly parametric pseudodifferential operators and Atiyah–Patodi–Singer boundary problems, Invent. Math. 121 (1995) 481–529.
[11] L. Guillopé, Théorie spectrale de quelques variétés à bouts, Ann. Sci. École Norm. Sup. (4) 22 (1989) 137–160.
[12] Y. Lee, Mayer–Vietoris formula for the determinants of elliptic operators of Laplace–Beltrami type (after Burghelea, Friedlander and Kappeler), Differ. Geom. Appl. 7 (1997) 325–340.
[13] Y. Lee, Burghelea–Friedlander–Kappeler's gluing formula for the zeta determinant and its applications to the adiabatic decompositions of the zeta-determinant and the analytic torsion, Trans. Amer. Math. Soc. 355 (10) (2003) 4093–4110.
[14] Y. Lee, Burghelea–Friedlander–Kappeler's gluing formula and the adiabatic decomposition of the zeta-determinant of a Dirac Laplacian, Manuscripta Math. 111 (2003) 241–259.
[15] Y. Lee, Asymptotic expansion of the zeta-determinant of an invertible Laplacian on a stretched manifold, Contemp. Math. 366 (2005) 95–108.
[16] P. Loya, J. Park, Decomposition of the ζ-determinant for the Laplacian on manifolds with cylindrical end, Illinois J. Math. 48 (2004) 1279–1303.
[17] P. Loya, J. Park, The ζ-determinant of generalized APS boundary problems over the cylinder, J. Phys. A: Math. Gen. 37 (2004) 7381–7392.
[18] P. Loya, J. Park, On gluing formulas for the spectral invariants of Dirac type operators, Electron. Res. Announc. Amer. Math. Soc. 11 (2005) 1–11.
[19] P. Loya, J. Park, On the gluing problem for the spectral invariants of Dirac operators, Adv. Math. 202 (2006) 401–450.
[20] W. Magnus, F. Oberhettinger, R.P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, Berlin, Heidelberg, New York, 1966.
[21] W. Müller, Eta invariant and manifolds with boundary, J. Differential Geom. 40 (1994) 311–377.
[22] W. Müller, Relative zeta functions, relative determinants and scattering theory, Comm. Math. Phys. 192 (1998) 309–347.

[23] J. Müller, M. Müller, Regularized determinants of Laplace type operators, analytic surgery and relative determinants, Duke Math. J. 133 (2006) 259–312.
[24] S.G. Scott, Determinants of Dirac boundary value problems over odd-dimensional manifolds, Comm. Math. Phys. 173 (1995) 43–76.
[25] S.G. Scott, Zeta determinants on manifolds with boundary, J. Funct. Anal. 192 (2002) 112–185.
[26] S.G. Scott, K.P. Wojciechowski, The ζ-determinant and Quillen determinant for a Dirac operator on a manifold with boundary, Geom. Funct. Anal. 10 (2000) 1202–1236.
[27] W. Voros, Spectral function, special functions and Selberg zeta function, Comm. Math. Phys. 110 (1987) 439–465.
[28] K.P. Wojciechowski, The ζ-determinant and the additivity of the η-invariant on the smooth, self-adjoint Grassmannian, Comm. Math. Phys. 201 (2) (1999) 423–444.